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REHEATING AND THERMALIZATION: LINEAR VS. NON-LINEAR RELAXATION

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Abstract

We consider the case of a scalar field, the inflaton, coupled to both lighter scalars and fermions, and the study the relaxation of the inflaton via particle production in both the linear and non-linear regimes. This has an immediate application to the reheating problem in inflationary universe models. The linear regime analysis offers a rationale for the standard approach to the reheating problem, but we make a distinction between relaxation and thermalization. We find that particle production when the inflaton starts in the *non-linear* region is typically a far more efficient way of transfering energy out of the inflaton zero mode and into the quanta of the lighter scalar than single particle decay. For the non-linear regime we take into account self-consistently the evolution of the expectation value of the inflaton field coupled to the evolution of the quantum fluctuations. An exhaustive numerical analysis reveals that the distribution of produced particles is far from thermal and the effect of open channels. In the fermionic case, Pauli blocking begins to hinder the transfer of energy into the fermion modes very early on in the evolution of the inflaton. We examine the implications of our results to the question of how to calculate the reheating temperature of the universe after inflation.

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I. INTRODUCTION

Thermalization, reheating and relaxation towards equilibrium are ubiquitous non-equilibrium phenomena that play a very important rôle in the early universe, in heavy ion collisions and in the dynamics of phase transitions away from equilibrium.

In particular in typical inflationary scenarios of the early universe, after the many e-folds of inflation necessary to solve the horizon and homogeneity problems, the matter and radiation energy density had been red-shifted to almost zero. However, once inflation has ended, this scenario has to merge with the standard big bang, radiation dominated cosmology. The succesful intertwining of inflation with the standard big bang cosmology thus necessitates some source of energy and entropy to rethermalize, or reheat the universe [1–3].

The standard picture [2–4] of old and new inflation invokes a scalar field, the inflaton, that produces a phase transition (either second or first order) and that at the end of inflation the expectation value of its zero momentum mode oscillates about the minimum of its (effective) potential. In chaotic inflation [5], the inflaton energy density still drives an inflationary phase, but there need not be any phase transition for this to occur. This inflaton field is constrained [1–3] to have a mass a few orders of magnitude smaller than the Planck mass, but typically much larger than the masses of the particles involved in the particle physics models. In the standard view of inflationary models, the expectation value of the inflaton field oscillates around the minimum of its potential, and its couplings to the lighter particles allow the inflaton field to decay. This decay process is then supposed to induce a damping term in the evolution equation for the inflaton expectation value of the form $\Gamma\dot{\phi}$ with Γ being the total decay rate of the inflaton field [2,6,7]. The standard estimate for the reheating temperature based on single-particle decay [2] is then obtained by comparing the total decay rate of the particle (Γ) to that of the expansion, obtaining $T_r \simeq 0.1\sqrt{\Gamma M_{pl}}$ [2].

Recent investigations of the non-linear quantum dynamics of scalar fields reveal a variety of new and striking phenomena [8–14]. The main relevant implication for the reheating problem being that the particle production induced by the time evolution of the inflaton is significatively **different** from linear estimates.

The non-linear (quantum) effects lead to an extremely effective dissipational dynamics and particle production even in the simplest self-interacting scalar field theory [11,13] in which single particle decay is kinematically forbidden. In the case in which only the *classical* evolution of the expectation value is taken into account in the evolution of the quantum fluctuations, this mechanism corresponds to parametric amplification or parametric resonance, because the classical time evolution is periodic in time [15]. However when back-reaction effects are taken into account, particle production damps out the evolution of the scalar field, and this damped evolution is incorporated in the equations for the quantum fluctuations. In this case the time evolution of the scalar field *including* the dissipative effects is not periodic; this implies that the situation is far different from parametric amplification. We refer to this situation, which includes the back-reaction effects as **induced amplification**, to distinguish it from parametric resonant amplification [15].

Back-reaction effects [11,13] drastically change the picture of particle production. In the case of parametric amplification, particle production never shuts off and the total number of particles created is infinite in Minkowski. It is only when the back-reaction is incorporated

that the damping effects on the expectation value feed back to the particle production mechanism, eventually shutting it off. This has been studied in detail analytically and numerically in [13] for a self-interacting scalar field.

Induced amplification is most effective when the amplitude of the expectation value of the scalar field (from the equilibrium value) is large and is, therefore, an intrinsically non-linear process, which we refer to as *non-linear relaxation*.

Particle production via parametric amplification of quantum fluctuations has been studied recently in the semiclassical approximation, but without back-reaction effects, in connection with fermion production by (pseudo) Nambu-Goldstone bosons [16].

Induced amplification and non-linear relaxation will be the primary mechanism for dissipation via particle production in the case of large amplitude of the scalar field. Such is the case in chaotic infation scenarios [5], as well as in the out of equilibrium regime during phase transitions for fields with very flat potentials near the maximum, which is typically required for a long inflationary phase and also occurs for the moduli fields in string theory [17].

As anticipated by Kofman, Linde and Starobinsky [11], this mechanism will necessarily modify the standard picture of reheating that was primarily based on the premise that particle production only ocurred when the inflaton oscillates with small amplitude at the bottom of the (effective) potential. Understanding the time scales and non-linear processes of reheating and eventually thermalization acquires further importance with the possibility that asymptotic oscillations of the inflaton field around the minima of the (effective) potential may still be present in today's universe in the form of dark matter [11,18].

Thermalization is a process that is fundamentally different from particle production. Typically, as will be seen below, the particles produced by the process of induced amplification will be in a non-thermal distribution [13]. Thermalization is a collisional process, in which the produced particles will exchange energy (and momentum) and eventually achieve a thermal distribution. In principle the time scales for dissipation via particle production may be widely different from the time scales of collisional thermalization. For very weakly coupled theories, if the distribution of produced particles via induced amplification is very far from thermal, many collisions will be necessary to thermalize the system and the time scale for thermalization may very well be large compared to the time scale of dissipation via particle production. This will have implications in cosmology discussed in the conclusions.

The process of thermalization, reheating and relaxation of perturbations, and the ensuing production of entropy is also very relevant in heavy ion collisions [19] and in phase transitions in particle physics, at the electroweak scale within the context of baryogenesis [20], as well as for the quark-gluon plasma, deconfining and chiral phase transitions [19,21].

Our main goal in this article is to study the time scales associated with the process of non-linear relaxation and compare these to those of single particle decay which we refer to as "linear relaxation".

In the linear relaxation case the important scale is determined by the decay rate Γ, usually referred to as the "damping rate". The literature [22] usually identifies $\Gamma \approx \mathrm{Im}\,\Sigma(\omega,\vec{k})/\omega$ with the "thermalization rate". Here $\mathrm{Im}\,\Sigma(\omega,\vec{k})$ is the imaginary part of the self-energy. In this article we offer a *real time* critique of this relation by studying the real time evolution and relaxation of linearized perturbations, and point out the following observations:

1. The damping rate is identical to the imaginary part of the self energy *only* when the inflaton (the particle interpolated by the order parameter) is a resonance with Γ being its width. In the complex frequency plane this corresponds to a pole in the second (unphysical) Riemann sheet. Even in this case, this damping rate corresponds to the relaxation of the *expectation value of the scalar field* and is exponential for some time regime. However, eventually relaxation continues with a power law tail. When the imaginary part is zero on-shell and the one particle pole is below the multiparticle thresholds, relaxation is given by a power law and no "damping rate" can be associated with the imaginary part of the self energy. Moreover, the relation $\Gamma \approx \text{Im}\,\Sigma(\omega, \vec{k})/\omega$ only holds in the linear approximation around equilibrium.

2. Even when relaxation of the expectation value of the scalar field is exponential, this damping rate determines the approach to equilibrium of the *expectation value* of the scalar field but it cannot be immediately inferred that the same time scale describes thermalization, i.e. the approach to a Bose Einstein distribution of an initial off-equilibrium distribution. Thermalization is a rather different process and has to be described with a Boltzmann equation with a collision term. Thus, rather than interpreting this time scale as a thermalization scale, we interpret it as a relaxation scale for the expectation value.

3. We consider the non-linear quantum field evolution (non-linear relaxation) incorporating the order parameter into the effective mass in a self-consistent way. [The self-consistency requirement makes the evolution equations non-linear]. We study the inflaton evolution in real time coupled both to light scalars and fermions, providing exhaustive numerical results. We find that the non-linear relaxation time scales can be **much shorter** than those predicted by linear relaxation. In addition, the particles are produced with a momentum distribution that is very far from thermal and skewed towards low momentum. In the case of fermions we provide numerical evidence for the phenomenon of Pauli blocking that hinders dissipation and production of fermion-antifermion pairs. This phenomenon is very similar to that found by Kluger and collaborators [23] in their studies of fermionic back-reaction in the presence of strong electric fields.

Both time scales will have to be understood in detail to provide a *quantitative* and reliable estimate of the reheating temperature.

Although we are ultimately interested in describing reheating and thermalization in Friedmann-Robertson-Walker cosmologies, in this article we will work in Minkowski space *assuming* that the relaxation time scales are much shorter than the expansion time scale. Furthermore, we want to isolate dissipative effects arising from particle production from those resulting from the red-shift in an expanding cosmology.

A short account of the results presented in this paper was reported in ref. [24].

In section II, we introduce the general model that we propose to study and summarize the necessary ingredients of non-equilibrium field theory to provide the reader with the essentials needed to reproduce our calculations, as these do not seem to be part of the standard techniques. In section III, we study linear relaxation in real time and elucidate the rôle of the imaginary parts of self energies, their interpretation in real time and criticize

the identification with "thermalization rates". In section IV we study the process of non-linear relaxation, and particle production both for scalars and fermions, providing exhaustive numerical results. Finally we conclude in section V with a discussion of the implications of our results and provide a guide to obtain estimates for the reheating temperature in model theories. An Appendix is devoted to a pedagogical exercise in non-equilibrium field theory.

II. THE MODEL AND THE METHODS

We consider the simplest model [2,3] where the inflaton field Φ couples to a scalar σ and to a fermion field ψ. That is,

$$\mathcal{L} = -\frac{1}{2}\Phi\left(\partial^2 + m_\Phi^2 + g\sigma^2\right)\Phi - \frac{\lambda_\Phi}{4!}\Phi^4 - \frac{1}{2}\sigma\left(\partial^2 + m_\sigma^2\right)\sigma - \frac{\lambda_\sigma}{4!}\sigma^4 + \bar{\psi}(i\partial\!\!\!/ - m_\psi - y\Phi)\psi \; . \tag{2.1}$$

The case $m_\sigma, m_\psi \ll m_\Phi$ will be of particular relevance, since in this case there are open decay channels for the inflaton.

We will investigate the scalar and fermionic couplings independently. Although the situation for reheating corresponds to (almost) zero temperature, we will study the case of linear relaxation at finite temperature. The reason for this is that finite temperature effects reflect the Bose enhancement and Pauli blocking factors that appear whenever there are excitations in the medium. These contributions from the medium will allow us to identify similar physical features in the case of non-linear relaxation.

A. Linear Relaxation: Amplitude and Perturbative expansion

To explore the behavior of the inflaton within the linear regime, we first use the tadpole method to obtain the equation of motion (A3) (see the Appendix for details on how this method is actually applied and reference [25] for an alternative implementation). The next step is to linearize this equation in the inflaton zero mode amplitude and use this to study the relaxational dynamics of the inflaton. We should note that this amplitude expansion is *a priori* different from the standard perturbative expansion in the relevant coupling constant. Later in this work, we will compare the results obtained here with results obtained through a self-consistent, non-perturbative resummation both in the coupling constants *and* the field amplitude.

We arrange the initial conditions to be such that the fields Φ, σ, ψ start to interact at a time that we choose $t = 0$. This can be accomplished by making the coupling "constants" time dependent, i.e. zero for $t < 0$ and different from zero for $t > 0$. To perform the calculations, we will need the non-equilibrium Green's functions and Feynman rules. Since the non-equilibrium generating functionals involve a forward and backward time contour [8,22,26], the number of vertices is doubled. Those in which all the fields are on the forward branch (fields labeled by $(+)$) are the usual interaction vertices, while those in which the fields are on the backward branch (fields labeled by $(-)$) have the opposite sign. The combinatoric factors are the same as in usual field theory. The spatial Fourier transform of the necessary finite (initial) temperature propagators are:

- Bosonic Propagators

$$
\begin{aligned}
&G^{++}_k(t,t') = G^{>}_k(t,t')\Theta(t-t') + G^{<}_k(t,t')\Theta(t'-t) \; , \\
&G^{--}_k(t,t') = G^{>}_k(t,t')\Theta(t'-t) + G^{<}_k(t,t')\Theta(t-t') \; , \\
&G^{+-}_k(t,t') = -G^{<}_k(t,t') \; , \\
&G^{-+}_k(t,t') = -G^{>}_k(t,t') \; , \\
&G^{>}_k(t,t') = i\int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \; \langle \Phi(\vec{x},t)\Phi(\vec{0},t')\rangle \\
&= \frac{i}{2\omega_k}\left\{[1+n_b(\omega_k)]e^{-i\omega_k(t-t')} + n_b(\omega_k)e^{i\omega_k(t-t')}\right\} \; , \\
&G^{<}_k(t,t') = i\int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \; \langle \Phi(\vec{0},t')\Phi(\vec{x},t)\rangle \\
&= \frac{i}{2\omega_k}\left\{[1+n_b(\omega_k)]e^{i\omega_k(t-t')} + n_b(\omega_k)e^{-i\omega_k(t-t')}\right\} \; , \\
&\quad \omega_k = \sqrt{\vec{k}^2+m^2} \; , \qquad\qquad n_b(\omega_k) = \frac{1}{e^{\beta\omega_k}-1} \; ,
\end{aligned}
\tag{2.2}
$$

where m is the mass of the boson.

- Fermionic Propagators (Zero chemical potential)

$$
\begin{aligned}
&S^{++}_{\vec{k}}(t,t') = S^{>}_{\vec{k}}(t,t')\Theta(t-t') + S^{<}_{\vec{k}}(t,t')\Theta(t'-t) \; , \\
&S^{--}_{\vec{k}}(t,t') = S^{>}_{\vec{k}}(t,t')\Theta(t'-t) + S^{<}_{\vec{k}}(t,t')\Theta(t-t') \; , \\
&S^{+-}_{\vec{k}}(t,t') = -S^{<}_{\vec{k}}(t,t') \; , \\
&S^{-+}_{\vec{k}}(t,t') = -S^{>}_{\vec{k}}(t,t') \; , \\
&S^{>}_{\vec{k}}(t,t') = -i\int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \; \langle \psi(\vec{x},t)\bar{\psi}(\vec{0},t')\rangle \\
&\qquad = -\frac{i}{2\omega_k}\left[e^{-i\omega_k(t-t')}(\not{k}+m_\psi)(1-n_f(\omega_k)) + e^{i\omega_k(t-t')}\gamma_0(\not{k}-m_\psi)\gamma_0 n_f(\omega_k)\right] \; , \\
&S^{<}_{\vec{k}}(t,t') = i\int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \; \langle \bar{\psi}(\vec{0},t')\psi(\vec{x},t)\rangle \\
&\qquad = \frac{i}{2\omega_k}\left[e^{-i\omega_k(t-t')}(\not{k}+m_\psi)n_f(\omega_k) + e^{-i\omega_k(t-t')}\gamma_0(\not{k}-m_\psi)\gamma_0(1-n_f(\omega_k))\right] \; , \\
&\quad \omega_k = \sqrt{\vec{k}^2+m_\psi^2} \; , \qquad\qquad n_f(\omega_k) = \frac{1}{e^{\beta\omega_k}+1} \; .
\end{aligned}
\tag{2.3}
$$

In the linear amplitude approximation, corresponding to linear relaxation, we find in all cases the following form of the equation of motion for the expectation value (see Appendix)

$$
\begin{aligned}
&\ddot{\delta}_{\vec{p}}(t) + \Omega^2_{\vec{p}}\,\delta_{\vec{p}}(t) + \int_0^\infty K_{\vec{p}}(t-t')\;\delta_{\vec{p}}(t')\;dt' = 0 \; , \\
&K_{\vec{p}}(t-t') = \Sigma_{r,\vec{p}}(t-t')\Theta(t-t') \; ,
\end{aligned}
\tag{2.4}
$$

where we have imposed as boundary conditions that the inflaton and the other fields are coupled at time $t=0$ but uncoupled for previous times, and introduced

$$\delta_{\vec{p}}(t) = \int d^3x \; e^{-i\vec{p}\cdot\vec{x}} \; \phi(\vec{x},t) \; , \quad \Omega^2_{\vec{p}} = \vec{p}^{\,2} + m^2_{\Phi} + \delta m(T) \; , \tag{2.5}$$

where $\delta m(T)$ is the time independent (but temperature dependent) contribution from tadpole diagrams. These contributions renormalize the mass and introduce a temperature dependent effective mass and will be specified later in each particular case. The quantity $\Sigma_{r,\vec{p}}(t-t')\Theta(t-t')$ is the retarded self-energy. It will be computed to dominant order in the couplings for both fermions and bosons. Although the self-energy at finite temperature has been computed before in the literature [27,28], we differ from previous treatments in that we perform our calculations directly in real time, this allows us to study real time relaxation as an initial condition problem.

Eq. (2.4) can be solved by Laplace transform. Define

$$\varphi_{\vec{p}}(s) = \int_0^{\infty} e^{-st} \; \delta_{\vec{p}}(t) \; dt \; . \tag{2.6}$$

Then, eq. (2.4) becomes

$$s^2 \varphi_{\vec{p}}(s) - s\delta_{\vec{p}}(0) - \dot{\delta}_{\vec{p}}(0) + \Omega^2_{\vec{p}} \; \varphi_{\vec{p}}(s) + \varphi_{\vec{p}}(s) \; \Sigma_{\vec{p}}(s) = 0 \; , \tag{2.7}$$

with $\Sigma_{\vec{p}}(s)$ the Laplace transform of $\Sigma_{r,\vec{p}}(t)$.

For computational purposes, it can be shown that at zero temperature each graph of $\Sigma_{\vec{p}}(s)$ is exactly equal to the corresponding graph of the zero-temperature equilibrium Euclidean quantum field theory, s being the time component of the Euclidean four momentum.

In general $\Sigma_{\vec{p}}(s)$ can be written as a dispersion integral in terms of the spectral density $\rho(p_o, \vec{p}, T)$

$$\Sigma_{\vec{p}}(s) = -\int \frac{2\,p_o\,\rho(p_o,\vec{p},T)}{s^2+p_o^2} dp_o \; . \tag{2.8}$$

The imaginary part of the self-energy is found to be

$$\begin{aligned} &\mathrm{Im}\,\Sigma_{\vec{p}}(s = i\omega \pm 0^+) = \pm \Sigma_{I\vec{p}}(\omega) \; , \\ &\Sigma_{I\vec{p}}(\omega) = \pi \; \mathrm{sign}\,(\omega) \left[\rho(|\omega|,\vec{p},T) - \rho(-|\omega|,\vec{p},T) \right] \; . \end{aligned} \tag{2.9}$$

The presence of sign (ω) in the above expression characterizes the *retarded* self-energy.

Let us choose $\delta_{\vec{p}}(0) = \delta_i \; , \dot{\delta}_{\vec{p}}(0) = 0$ for simplicity. We get from eq. (2.7)

$$\varphi_{\vec{p}}(s) = \delta_i \; \frac{s}{s^2 + \Omega^2_{\vec{p}} + \Sigma_{\vec{p}}(s)} \; . \tag{2.10}$$

The Laplace transform can be inverted through the formula

$$\delta_{\vec{p}}(t) = \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} e^{st} \; \varphi_{\vec{p}}(s) \; \frac{ds}{2\pi i} \; , \tag{2.11}$$

where ϵ is a positive real constant (Bromwich countour). Thus we need to understand the singularities of $[s^2 + \Omega^2_{\vec{p}} + \Sigma_{\vec{p}}(s)]^{-1}$. We now consider the following cases:

- The inflaton potential admits spontaneous symmetry breaking (SSB), and is only coupled to lighter *scalar* fields.

- The inflaton is coupled to fermions only.

We will also consider the subcases where the initial temperature is taken to be zero, as would be appropriate in the case of evolution in the post inflationary universe, as well as the situation where the initial temperature is non-zero, which would be relevant to the situation of a scalar field starting in an initial (non-equilibrium) but thermal state and evolving out of it.

B. SSB with Coupling to Lighter Scalars Only

If $m_\Phi^2 = -\mu^2 < 0$, then the new minimum is at $\Phi_0 = \sqrt{6\mu^2/\lambda_\Phi}$, and we write $\Phi^\pm(\vec{x},t) = \Phi_0 + \phi(\vec{x},t) + \chi^\pm(\vec{x},t)$. The masses are now shifted to

$$M^2 = 2\mu^2 \; , \qquad\qquad M_\sigma^2 = m_\sigma^2 + g^2\Phi_0^2 \; . \tag{2.12}$$

The contribution from the quartic inflaton self-coupling has been studied previously [13] thus it will not be repeated here.

The tadpole correction to the mass in eq. (2.5) is given by

$$\begin{aligned} &\delta M(T) = -ig\int \frac{d^3k}{(2\pi)^3} G^{++}_{k,\sigma}(t,t) = g\int \frac{d^3k}{(2\pi)^3}\frac{1+2\,n_b(\omega_k)}{2\omega_k} \; , \\ &\omega_k = \sqrt{\vec{k}^2 + M_\sigma^2} \; . \end{aligned}$$

This mass renormalization does not influence the dynamics. The retarded self energy is found to be at order g^2,

$$K_{\vec{p}}(t-t') = 2ig^2\Phi_0^2\int\frac{d^3k}{(2\pi)^3}\left[G^{++}_{\vec{k},\sigma}(t-t')G^{++}_{\vec{k}+\vec{p},\sigma}(t-t') - G^{<}_{\vec{k},\sigma}(t-t')G^{<}_{\vec{k}+\vec{p},\sigma}(t-t')\right] . \tag{2.13}$$

With the non-equilibrium Green's functions defined above, we find

$$\begin{aligned} \Sigma_{r,\vec{p}}(t-t') = -2g^2\Phi_0^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}\omega_{\vec{k}+\vec{p}}}&\Big\{[1+2\,n_b(\omega_k)]\sin[(\omega_{\vec{k}}+\omega_{\vec{k}+\vec{p}})(t-t')] \\ &-2\,n_b(\omega_k)\sin[(\omega_{\vec{k}}-\omega_{\vec{k}+\vec{p}})(t-t')]\Big\} \; . \end{aligned} \tag{2.14}$$

The Laplace transform can be written as a dispersion integral in terms of the bosonic spectral density $\rho_b(p_o,\vec{p},T)$ (see eq. (2.8)). We have to one loop level:

$$\begin{aligned} \rho_b(p_o,\vec{p},T) = 2g^2\Phi_0^2 &\int\frac{d^3k}{(2\pi)^3 2\omega_k}\int\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}(2\pi)^3\delta^3(\vec{p}-\vec{k}-\vec{k}') \\ &\times\left[\delta(p_o-\omega_k-\omega_{k'})(1+\,2\,n_b(\omega_k)) - \delta(p_o-\omega_k+\omega_{k'})\,2\,n_b(\omega_k)\right] \; . \end{aligned}$$

The imaginary part of the self-energy is given by eq. (2.9).

We will analyze only the case of a spatially constant order parameter corresponding to $\vec{p}=0$ because we will later compare with the case of non-linear relaxation which we only study for a homogeneous (translational invariant) expectation value. In this case the spectral density can be written as

$$\rho_b(p_o,\vec{0},T) = \left[1+2\,n_b\left(\frac{p_o}{2}\right)\right]\rho_b(p_o,\vec{0},0)\;. \tag{2.15}$$

The spectral density $\rho_b(p_o,\vec{0},0)$ is a Lorentz scalar and is proportional to the decay rate of the boson Φ into two σ particles. It is a straightforward exercise to find

$$\rho_b(p_o,\vec{0},0) = \frac{g^2\Phi_0^2}{8\pi^2}\left[1-\frac{4M_\sigma^2}{p_o^2}\right]^{\frac{1}{2}}\Theta\left(p_o^2-4M_\sigma^2\right)\;. \tag{2.16}$$

Clearly $\Sigma_{\vec{0}}(s)$ has a logarithmic divergence, independent of s and T. We choose to subtract this divergence at $s=0$ and absorb the subtraction into a further temperature dependent renormalization of M. In order not to clutter notation, from now on we will refer to M as the fully renormalized mass including the above subtraction, and to $\Sigma(s)\equiv\Sigma_{\vec{0}}(s)-\Sigma_{\vec{0}}(0)$. From equations (2.8, 2.15, 2.16) we find that the self-energy has an imaginary part above the two σ-particles threshold given by eq. (2.9) with

$$\Sigma_I(\omega) = \frac{g^2\Phi_0^2}{8\pi}\left[1-\frac{4M_\sigma^2}{\omega^2}\right]^{\frac{1}{2}}\left[1+2\,n_b\left(\frac{\omega}{2}\right)\right]\Theta\left(\omega^2-4M_\sigma^2\right)\;\text{sign}\;(\omega)\;. \tag{2.17}$$

This expression is recognized as the imaginary part of the retarded self-energy (determined by the sign(ω)) and shows the usual Bose enhancement factor [29].

1. Zero Temperature

In this case from 2.14 we find

$$\Sigma(s) = \frac{g^2\Phi_0^2}{4\pi^2}\left(\sqrt{1+\frac{4M_\sigma^2}{s^2}}\;\text{ArgTh}\,\frac{1}{\sqrt{1+\frac{4M_\sigma^2}{s^2}}}-1\right)\;. \tag{2.18}$$

In order to compute the inverse Laplace transform through eq. (2.11) we must first study the analytic structure of $\varphi(s)$ in the s-plane. $\varphi(s)$ has poles at the zeroes of

$$s^2+M^2+\Sigma(s)=0\;. \tag{2.19}$$

These correspond to Φ-one-particle states with the mass including one-loop radiative corrections. At zeroth order the poles are purely imaginary

$$s_\pm = \pm iM\;. \tag{2.20}$$

To find the one-loop correction, we set

$$s_+ = iM + r \tag{2.21}$$

and similarly for s_-. Inserting this in eq. (2.19) yields to order g^2

$$2iMr+\Sigma(iM)=0\;. \tag{2.22}$$

That is,

$$r = i\frac{\Sigma(iM)}{2M} \ . \tag{2.23}$$

When $M < 2M_\sigma$, $\Sigma(iM)$ is real (see eq. (2.18)) and eq. (2.23) gives a real correction to the Φ mass

$$s_\pm = \pm iM_0 \equiv \pm i\left[M + \frac{g^2\Phi_0^2}{8\pi^2 M}\left(\sqrt{\frac{4M_\sigma^2}{M^2}-1}\ \ \mathrm{ArgTh}\,\frac{1}{\sqrt{\frac{4M_\sigma^2}{M^2}-1}} - 1\right)\right] \ . \tag{2.24}$$

The Laplace transform $\varphi(s)$ also exibits a cut along the imaginary axis starting at $s = i\omega = \pm 2iM_\sigma$. For s in the first Riemann sheet (physical sheet) we obtain

$$\Sigma_{\mathrm{physical}}(i\omega \pm 0^+) = \Sigma_R(\omega) \pm i\Sigma_I(\omega) \ , \qquad \omega > 2M_\sigma \ , \tag{2.25}$$

with Σ_R and Σ_I both real and given by

$$\Sigma_R(\omega) = \frac{g^2\Phi_0^2}{4\pi^2}\left(\sqrt{1-\frac{4M_\sigma^2}{\omega^2}}\ \ \mathrm{ArgTh}\sqrt{1-\frac{4M_\sigma^2}{\omega^2}} - 1\right) \ , \qquad \Sigma_I(\omega) = \frac{g^2\Phi_0^2}{8\pi}\sqrt{1-\frac{4M_\sigma^2}{\omega^2}} > 0 \ . \tag{2.26}$$

We can now proceed to compute the inverse Laplace transform (2.11) by deforming the contour.

$$\delta(t) = \frac{\delta_i \cos M_0 t}{1 - \frac{\partial\Sigma(iM)}{\partial M^2}} + \frac{2\delta_i}{\pi}\int_{2M_\sigma}^{\infty} \frac{\omega\Sigma_I(\omega)\cos\omega t \; d\omega}{[\omega^2 - M^2 - \Sigma_R(\omega)]^2 + \Sigma_I(\omega)^2} \ . \tag{2.27}$$

For large time $M_\sigma t \gg 1$ the integral over the cut is dominated by the endpoint $\omega = 2M_\sigma$ and goes to zero as

$$\delta_{\mathrm{cut}}(t) \simeq \frac{\delta_i \sqrt{\pi}\ M_\sigma^2\ g^2\Phi_0^2}{4\pi^2(M^2 - 4M_\sigma^2 + \frac{g^2\Phi_0^2}{4\pi^2})^2}\frac{\cos(2M_\sigma t + \frac{3\pi}{4})}{(M_\sigma t)^{\frac{3}{2}}} \ . \tag{2.28}$$

The $t^{-3/2}$ is completely determined by the behavior of the spectral density at threshold.

The situation changes drastically for $M > 2M_\sigma$. In such case the Φ-particle is unstable and thus $\Sigma(iM)$ becomes complex and its value depends from which side of the cut we approach the imaginary axis. Now the solution for the pole will be complex and r will acquire a *real* part. Due to the discontinuity in Σ_I across the two-particle cut, eq. (2.23) must be written carefully as

$$r = \frac{i}{2M}\ \Sigma(iM + \mathrm{Re}\,[r]) \ , \tag{2.29}$$

where the (small) real correction inside the argument will determine on which side of the cut the solution resides. Then the real part of r should satisfy

$$\mathrm{Re}\, r = -\frac{1}{2M}\,\mathrm{Im}\,\Sigma(iM + 0^+ \ \mathrm{sign}\,(\mathrm{Re}\,[r])) \ . \tag{2.30}$$

From eq. (2.17) we see that $\mathrm{Im}\,\Sigma_{\rm physical}$ is negative for $\mathrm{sign}\,(\mathrm{Re}\,[r]) < 0$ and positive for $\mathrm{sign}\,(\mathrm{Re}\,[r]) > 0$. Therefore eq. (2.30) *has no solution* in the physical sheet.

The analytic continuation of Σ into the second Riemann sheet is such that [30]

$$\Sigma^{II}(i\omega \pm 0^+) = \Sigma_R(\omega) \mp i\Sigma_I(\omega)\,, \qquad\qquad \omega > 2M_\sigma \tag{2.31}$$

and we find the solution

$$\begin{aligned}
\mathrm{Re}\,r &= -\frac{1}{2M}\Sigma_I(M) = -\frac{g^2\Phi_0^2}{16\pi M}\sqrt{1-\frac{4M_\sigma^2}{M^2}} < 0\,, \\
\mathrm{Im}\,r &= \frac{1}{2M}\Sigma_R(M) = \frac{g^2\Phi_0^2}{8\pi^2 M}\left(\sqrt{1-\frac{4M_\sigma^2}{M^2}}\ \mathrm{ArgTh}\sqrt{1-\frac{4M_\sigma^2}{M^2}} - 1\right)\,.
\end{aligned}$$

For $M \gg M_\sigma$,

$$\mathrm{Im}\,r = \frac{g^2\Phi_0^2}{8\pi^2 M}\left(\log\frac{M}{M_\sigma} - 1\right)\,. \tag{2.32}$$

$|\mathrm{Re}\,[r]|$ coincides with the decay rate $\Phi \to 2\sigma$ (as it must be) which is the rate per unit time to produce σ particles. The poles $s_\pm$ move off into the second sheet when M becomes larger than $2M_\sigma$ as expected [30].

Thus when we compute the inverse Laplace transform (2.11) in the unstable case we are left with the integral over the cuts since both poles are in the second Riemann sheet and we find

$$\delta(t) = \frac{2\delta_i}{\pi}\int_{2M_\sigma}^{\infty}\frac{\omega\Sigma_I(\omega)\cos\omega t\, d\omega}{[\omega^2 - M^2 - \Sigma_R(\omega)]^2 + \Sigma_I(\omega)^2}\,. \tag{2.33}$$

Since now M is inside the integration region, for weak coupling there is a narrow resonance at $\omega \simeq M$. Thus for weak coupling it takes Breit-Wigner form and we find to a very good approximation,

$$\delta(t) \simeq \delta_i\, A\, e^{-\Gamma t/2}\ \cos(Mt + \alpha)\,, \qquad\qquad \Gamma \ll M\,, \tag{2.34}$$

where

$$A = 1 + \frac{\partial\Sigma_R(M)}{\partial M^2}\,, \qquad \Gamma = \frac{g^2\Phi_0^2}{8\pi M}\sqrt{1-\frac{4M_\sigma^2}{M^2}}\,, \qquad \alpha = -\frac{\partial\Sigma_I(M)}{\partial M^2}\,. \tag{2.35}$$

For $M \gg M_\sigma$,

$$A = 1 + \frac{g^2\Phi_0^2}{8\pi^2 M^2} + \mathcal{O}\left(\left[\frac{M_\sigma}{M}\right]^2\right)\,, \qquad\qquad \alpha = \frac{g^2\Phi_0^2 M_\sigma^2}{4\pi M^4} + \mathcal{O}\left(\left[\frac{M_\sigma}{M}\right]^4\right)\,. \tag{2.36}$$

The Breit-Wigner approximation, however, is valid only for times $\leq \Gamma^{-1}\ln(\Gamma/M_\sigma)$; for longer times the fall off is with a power law $t^{-3/2}$ determined by the spectral density at threshold as before.

2. Non-Zero Temperature

The physical mass gets a finite temperature dependent correction from the tadpole contribution $\delta M(T)$ given by

$$\delta M(T) = g \int_0^\infty \frac{dk}{2\pi^2} \frac{k^2}{\sqrt{k^2+M_\sigma^2}} \frac{1}{e^{\beta\sqrt{k^2+M_\sigma^2}}-1} \; . \tag{2.37}$$

For small β ($T \gg M_\sigma$) this correction takes the form

$$\delta M(T) = g \left\{ \frac{T}{12} - \frac{M_\sigma T}{4\pi} + \frac{M_\sigma^2}{8\pi} \log(\frac{T}{M_\sigma}) + \mathcal{O}(T^0) \right\} \; . \tag{2.38}$$

We will assume that $gT^2 \ll M^2$ so that we are in the perturbative regime, since otherwise hard thermal loops must be resummed [31,32], a task beyond the scope of this article.

The imaginary part of the self-energy is given in eq. (2.17) and the real part can be obtained from the dispersion integral (2.8) using (2.15, 2.16). The real part of the self-energy is difficult to compute for arbitrary temperature, and we just quote its large T behavior:

$$\Sigma(s,\beta) = g^2 \Phi_0^2 \frac{M_\sigma T}{\pi s^2} \left[1 - \sqrt{1 + \frac{s^2}{4M_\sigma^2}} \right] + \mathcal{O}(T^0) \; . \tag{2.39}$$

It is interesting to compute $\Sigma(t,\beta)$ in configuration space for large T. We have by inverse Laplace transform

$$\Sigma(t,\beta) = \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} e^{st}\, \Sigma(s,\beta)\, \frac{ds}{2\pi i} \; . \tag{2.40}$$

Upon contour deformation we find

$$\Sigma(t,\beta) = -\frac{g^2\Phi_0^2 T}{\pi^2} \int_1^\infty \frac{dx}{x} \, \sqrt{x^2-1} \, \sin(2M_\sigma x t) \; . \tag{2.41}$$

This function is obviously **not** concentrated at $t=0$. It can be related to a J_1 Bessel function as

$$\Sigma(t,\beta) = -\frac{g^2\Phi_0^2 T}{2\pi} \left[1 - \frac{4M_\sigma}{\pi}\, t - 2\, M_\sigma \int_0^t dx \, \left(\frac{t}{x} - 1 \right) \, J_1(2\, M_\sigma x) \right] \tag{2.42}$$

We find for small and for large t

$$\Sigma(t,\beta) \overset{t\to 0}{=} -\frac{g^2\Phi_0^2 T}{2\pi} \left[1 - \frac{4M_\sigma}{\pi}\, t + O(t^2) \right]$$

$$\Sigma(t,\beta) \overset{t\to\infty}{=} \frac{g^2\Phi_0^2 T}{2\pi^2} \frac{\sin(2M_\sigma t - \pi/4)}{(M_\sigma t)^{3/2}} \left[1 + O(t^{-1}) \right] \; .$$

The behaviour of the kernel $\Sigma(t,\beta)$ shows that it **cannot** be approximated by a phenomenological term $\Gamma \frac{d}{dt}$ even for high T. (As shown in ref. [13] this is not the case either for $T=0$).

We can now repeat the analysis of the previous section for both cases $M < 2M_\sigma$ and $M > 2M_\sigma$. In the first case, the one particle pole is below the two σ-particles threshold with a (small) finite temperature correction since we are restricted to the perturbative regime in which $gT^2 \ll M$. The long time behavior will be oscillatory with the frequency corresponding to the one particle pole plus long-time power law tails similar to the zero temperature case. The second case is more interesting, since now the scalar "inflaton" is unstable, and the pole moves off into the second Riemann sheet. In the physical sheet there is a resonance with a finite temperature width given by

$$\Gamma(T) = \Gamma(0)\left[1 + 2\,n_b\left(\frac{M}{2}\right)\right] , \qquad \Gamma(0) = \frac{g^2\Phi_0^2}{8\pi M}\sqrt{1 - \frac{4M_\sigma^2}{M^2}} \; . \tag{2.43}$$

The Bose enhancement factor increases the rate and therefore enhances dissipation via the production of particles. Although this factor arises from the thermal distribution, we expect in general that whenever there are bosonic excitations present, the relaxation rate will be enhanced as a consequence of bose statistics, independently of whether these excitations are thermally distributed.

C. Unbroken symmetry with coupling to light scalars

In the unbroken symmetry case M_Φ is above all thresholds and hence Φ is a stable particle. The order parameter is again given by a formula like eq. (2.27) except that now the integration starts at $\omega = M + 2M_\sigma$.

The first perturbative contribution to the kernel $\Sigma(t - t')$ in the inflaton equation of motion (2.4) are now the two loop order graphs usually called "setting sun". Since these graphs are quite complicated, we only perform the two loop computation at zero temperature, for which we can exploit the relationship with usual Euclidean field theory. Even with this simplification the computation is hard, and we limit ourselves to the evaluation of the imaginary part of the retarded self-energy near the branch point. We have seen below eq. (2.27) how this actually determines the long time behaviour of the order parameter. In general, if $\Sigma_I(\omega \to \omega_{\mathrm{threshold}})$ vanishes as $(\omega - \omega_{\mathrm{threshold}})^\alpha$, then $\delta_{\mathrm{cut}}(t)$ decays as $t^{-1-\alpha}$ for large times.

We find for the two loops self-energy

$$\Sigma_I(\omega \to M + 2M_\sigma) \simeq \frac{2g^2\pi^2}{(4\pi)^4}\frac{M_\sigma\sqrt{M}}{(M+2M_\sigma)^{7/2}}[\omega^2 - (M+2M_\sigma)^2]^2 \; . \tag{2.44}$$

This yields for $\delta_{\mathrm{cut}}(t)$ a power decay t^{-3} for long times.

For massless σ we find

$$\Sigma_I(\omega \to M) \simeq \frac{g^2\pi}{3M^4(4\pi)^4}(\omega^2 - M^2)^3 \; , \tag{2.45}$$

yielding a $\delta_{\mathrm{cut}}(t)$ decaying as t^{-4} for long times. In summary, in the unbroken symmetry case $\delta(t)$ is given by the one-particle term oscillating with frequency M_0 plus the power damped cut contribution $\delta_{\mathrm{cut}}(t)$.

D. Inflaton coupled to Fermions

Fermions can be treated similarly in the broken and unbroken symmetry phase. To treat both cases on equal footing we now define M as mass of the inflaton (scalar) field and m as the fermion mass in either case. Using the Feynman rules described in the first section, we obtain to one-loop level the kernel

$$K_{\vec{p}}(t-t') = iy^2 \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}\, \left[iS^{++}_{\vec{k}}(t,t') iS^{++}_{\vec{k}-\vec{p}}(t',t) - iS^{<}_{\vec{k}}(t,t') iS^{>}_{\vec{k}-\vec{p}}(t',t) \right] ,$$

$$\Sigma_{r,\vec{p}}(t-t') = iy^2 \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}\, \left[iS^{>}_{\vec{k}}(t,t') iS^{<}_{\vec{k}-\vec{p}}(t',t) - iS^{<}_{\vec{k}}(t,t') iS^{>}_{\vec{k}-\vec{p}}(t',t) \right] .$$

We now concentrate on the homogeneous case $\vec{p}=0$. Using the fermionic Green's functions given in the first section it is straightforward to find the fermionic spectral density to be used in eq.(2.8)

$$\rho_f(p_o,T) = \frac{y^2}{8\pi^2}\, p_o^2 \left[1 - 2n_f\left(\frac{p_o}{2}\right)\right] \left[1 - \frac{4m^2}{p_o^2}\right]^{\frac{3}{2}} \Theta\left(p_o^2 - 4m^2\right) .$$

From eq.(2.9), the imaginary part of the self energy is then

$$\Sigma_I(\omega) = \frac{y^2}{8\pi}\omega^2 \left[1 - \frac{4m^2}{\omega^2}\right]^{\frac{3}{2}} \left[1 - 2n_f\left(\frac{\omega}{2}\right)\right] \Theta\left(\omega^2 - 4m^2\right) .$$

The finite temperature factor reflects the Pauli blocking term [29]. It is clear that the zero temperature part of $\Sigma(s)$ diverges quadratically and that two subtractions are needed. The first one is independent of s and contributes a (quadratically divergent) mass renormalization. The second one is logarithmic divergent and consists of an s independent term that adds to the mass renormalization and another proportional to s^2 that will be absorbed in wave function renormalization. We choose to subtract at zero temperature and at an arbitrary scale κ. The Laplace transform for the zero momentum component of the equation of motion (2.10) becomes

$$\varphi(s) = \frac{\delta_i\; s}{s^2\left(1 + \frac{y^2}{4\pi^2}\ln\frac{\Lambda}{\kappa}\right) + M^2_{1R}(T) + y^2\,\Pi\left(s,T,\kappa\right)} , \tag{2.46}$$

where M^2_{1R} contains the mass renormalization and Π is the twice subtracted kernel, which depends on the renormalization scale. Defining

$$\begin{aligned} Z^{-1}_\phi(\kappa) &= 1 + \frac{y^2}{4\pi^2}\ln\frac{\Lambda}{\kappa} , \\ y_R(\kappa) &= Z^{\frac{1}{2}}_\phi\; y , \\ M_R(T,\kappa) &= Z^{\frac{1}{2}}_\phi\; M_{1R}(T) , \\ \varphi_R(s,\kappa) &= Z^{-1}_\phi\; \varphi(s) , \end{aligned}$$

we finally obtain the renormalized Laplace transform

$$\varphi_R(s,\kappa) = \frac{\delta_i \; s}{s^2 + M_R^2(T,\kappa) + y_R^2(\kappa)\; \Pi\,(s,T,\kappa)} \; . \tag{2.47}$$

The inverse Laplace transform is obtained as in equation (2.11). The result will be the function $\delta_{R,\vec{0}}(t,\kappa)$ which is not a renormalization group invariant. It is clear, however, from the renormalization prescriptions described above that ratios of the amplitude at different times, such as $\delta_{R,\vec{0}}(t,\kappa)/\delta_{R,\vec{0}}(0,\kappa)$ are renormalization group invariant. Now the analysis can proceed as in the previous section. To obtain the inverse Laplace transform we must recognize the singularities in (2.47). If $M_R < 2m$ (m is the fermion mass in the loop), there is a one particle pole (with strength different from one because we decided to renormalize off-shell) and a two-fermion cut at $s^2 = -4m^2$. At long times the amplitude will oscillate with an oscillation period given by the position of the pole, which is perturbatively close to M_R. The contribution of the cut falls-off at long times as $t^{-5/2}$ and is determined by the behavior of the spectral density near the two-fermion threshold.

More interesting is the case in which $M_R > 2m$. As in the bosonic case, the pole moves off the physical sheet into the second sheet. In the physical sheet the spectral density at weak coupling features a sharp peak at $M_R + \mathcal{O}(y^2)$ with width

$$\Gamma(T) = \Gamma(0)\left[1 - 2n_f\left(\frac{M_R}{2}\right)\right] \; , \qquad \Gamma(0) = \frac{y^2}{8\pi} M_R \left[1 - \frac{4m^2}{M_R^2}\right]^{\frac{3}{2}} \; . \tag{2.48}$$

A Breit-Wigner approximation predicts exponential relaxation but eventually at long times (an estimate similar to the bosonic case) a power law relaxation $t^{-5/2}$ ensues, completely determined by the spectral density at threshold.

Pauli blocking makes the resonance narrower and the lifetime of the decaying particle longer. The interpretation of this phenomenon is simple. In the thermal bath, fermionic excited states are filled with the Fermi-Dirac distribution. In order for the scalar field to decay, it must create a fermion-antifermion pair. However, at finite temperature, the available states are already filled with thermal excitations and because of the Pauli exclusion principle, are not available. At infinite temperature (and zero chemical potential), each fermion and antifermion state are populated with occupation $1/2$ per spin degree of freedom; in this limit the decay rate goes to zero, the lifetime to infinity and the bosonic particle simply cannot decay because there are no states available to decay into. Even at zero temperature, but in a situation in which excited states are occupied, dissipative processes mediated by the production of fermion-antifermion pairs will be hindered by the Pauli exclusion principle, since states will already be occupied and no longer available in the particle production process. In highly excited states, we expect damping via production of fermion pairs to be strongly suppressed by Pauli blocking. This phenomenon has been seen numerically in the case of fermion pair production in strong electric fields [23] and will be seen numerically in the non-linear relaxation case later.

E. Thermalization or Relaxation?

From the analysis presented above, the following conclusions for the linear regime become very clear:

1. The imaginary part of the self-energy only determines a *relaxation rate* in the case of a **resonance**, that is when there is an imaginary part on-shell for the external particle and the (quasi) particle pole moves off the physical sheet into the second (unphysical) sheet. In this case a Breit-Wigner approximation describes the exponential relaxation for a long time, but eventually the amplitude falls-off with a power law in time, with the power determined by the behavior of the spectral density at threshold. When the on-shell pole is below the two-particle threshold, the imaginary part does not translate to a damping rate. Relaxation is described by a power law with an asymptotic behavior completely determined by the position and residue of the pole.

2. Even in the case of a resonance and exponential damping, the “damping rate” Γ describes exponential relaxation for the *expectation value* of the scalar field. The issue of thermalization is completely different. Thermalization corresponds to the time evolution of the (quasi) particle distribution function towards a thermal distribution which, in principle, has nothing to do with the relaxation of the expectation value of the field.

An alternative way to look at thermalization is as a process of momentum and energy transfer. Thus, the thermalization rate should be identified with the rate of energy and momentum transfer which is not necessarily related to the relaxation rate of the expectation value of the field. In order to understand thermalization, a collisional (quantum) Boltzmann equation must be set up. Although in the Born approximation the collision term includes the scattering cross section that is related to the decay rate, the solution to the Boltzmann equation implies a resummation that is quite different from the resummation of the Dyson series for the propagator. If the initial distribution is very far from thermal and many collisions are necessary for thermalization, the relaxation and thermalization time scales may be widely different and in principle unrelated. Thus we insist that the “damping rate” obtained from the imaginary part of the self-energy *on shell* must be interpreted as the relaxation rate for the expectation value of the scalar field and in principle *not* with the thermalization rate of the particle distribution function. Moreover, this relaxation rate holds in the linear regime (small field amplitude).

III. NON-LINEAR RELAXATION

In this section we study the equation of motion for a homogeneous order parameter beyond the linear approximation. This is achieved as follows [13]. We write the scalar inflaton field as

$$\Phi^{\pm}(\vec{x},t) = \phi(t) + \chi^{\pm}(\vec{x},t) \; , \tag{3.1}$$

with $\phi(t)$ the expectation value in the non-equilibrium density matrix and $< \chi^{\pm}(\vec{x},t) >= 0$, we consider that $< \sigma^{\pm} >= 0$ (this is a consistent assumption). The non-equilibrium path integral requires the Lagrangian density

$$\mathcal{L}(\phi;\chi^+;\sigma^+;\chi^-;\sigma^-) = \mathcal{L}(\phi;\chi^+;\sigma^+) - \mathcal{L}(\phi;\chi^-;\sigma^-) \; ,$$
$$\mathcal{L}(\phi;\chi^+;\sigma^+) = \frac{1}{2}\left[(\partial_\mu\chi^+)^2 - M^2(t)\,(\chi^+)^2\right] - \chi^+\left[\ddot{\phi} + m_\Phi^2\,\phi + \frac{\lambda}{6}\,\phi^3\right] - \frac{\lambda}{6}\phi\,(\chi^+)^3 - \frac{\lambda}{4!}\,(\chi^+)^4$$

$$+\frac{1}{2}\left[(\partial_\mu\sigma^+)^2 - m^2(t)\,(\sigma^+)^2\right] - g\,\phi\,\chi^+\,(\sigma^+)^2 - \frac{g}{2}\,(\chi^+)^2\,(\sigma^+)^2 + \bar{\psi}^+(i\,\not{\partial} - m_\psi(t) - y\chi^+)\psi^+ \; ,$$

$$M^2(t) = m_\Phi^2 + \frac{\lambda}{2}\,\phi^2(t) \; , \quad m^2(t) = m_\sigma^2 + g\,\phi^2(t) \; , \quad m_\psi(t) = m_\psi + y\,\phi(t) \; .$$

The difference with the linear relaxation case is that we now incorporate $\phi(t)$ in the definition of the time dependent masses. The equations of motion are obtained as in the previous sections, by treating the *linear*, cubic and quartic terms as perturbations. The necessary Green's functions are constructed from the homogeneous solutions of the quadratic forms. We again treat the cases where the inflaton is coupled to scalars or fermions separately.

A. Inflaton Coupled to Scalars Only

The Green's functions for the scalars are obtained from the mode equations that solve

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + M^2(t)\right] U_k(t) = 0 \; , \tag{3.2}$$

$$U_k(0) = 1 \; , \quad \dot{U}_k(0) = -iW_k = -i\sqrt{\vec{k}^2 + M^2(0)} \; ,$$

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2(t)\right] V_k(t) = 0 \; , \tag{3.3}$$

$$V_k(0) = 1 \; , \quad \dot{V}_k(0) = -i\,w_k = -i\sqrt{\vec{k}^2 + m^2(0)} \; ,$$

The initial conditions on the mode functions (3.2, 3.3) correspond to positive frequency solutions at the initial time. In terms of these mode functions, the Green's functions are [13,8]

$$G^{>}_{\chi,k}(t,t') = \frac{i}{2W_k}\left[(1+n_b(W_k))\;U_k(t)U_k^*(t') + n_b(W_k)\;U_k^*(t)U_k(t')\right] \; ,$$

$$G^{<}_{\chi,k}(t,t') = G^{>}_{\chi,k}(t',t) \; ,$$

$$G^{>}_{\sigma,k}(t,t') = \frac{i}{2w_k}\left[(1+n_b(w_k))\;V_k(t)V_k^*(t') + n_b(w_k)\;V_k^*(t)V_k(t')\right] \; ,$$

$$G^{<}_{\sigma,k}(t,t') = G^{>}_{\sigma,k}(t',t)$$

and the rest of the Green's functions are given by the relations in equations (2.2). The Green's functions above correspond to the situation in which the initial density matrix is in equilibrium for the positive and negative frequency modes at time $t = 0$. This is determined by the initial conditions on the mode functions above at an initial temperature $T = 1/\beta$. However, for the rest of the analysis we will take $T = 0$. We see that

$$< \chi^2(\vec{x},t) > = \int \frac{d^3k}{(2\pi)^3}\frac{|U_k(t)|^2}{2W_k}\coth\left[\frac{\beta W_k}{2}\right] \; ,$$

$$< \sigma^2(\vec{x},t) > = \int \frac{d^3k}{(2\pi)^3}\frac{|V_k(t)|^2}{2w_k}\coth\left[\frac{\beta w_k}{2}\right] \; .$$

Finally the equation of motion to one-loop order for the expectation value is

$$\ddot{\phi}(t) + m_\Phi^2 \, \phi(t) + \frac{\lambda}{6} \, \phi^3(t) + \frac{\lambda}{2} \, \phi(t) < \chi^2(\vec{x},t) > + g \, \phi(t) \; < \sigma^2(\vec{x},t) >= 0 \tag{3.4}$$

If one wants to solve the equation of motion (3.4) to order ($\hbar$) one would expand ϕ in a power series in $\hbar$ and only keep the zeroth order term in the mode equations (3.2, 3.3). As was observed previously [13], such an expansion will result in secular terms and becomes unreliable at long times. A resummation is necessary to capture the long time behavior consistently. We will perform a non-perturbative resummation of the one-loop terms by solving the set of equations (3.2 - 3.4) with the *full* value of ϕ in the mode equations. This then becomes a set of coupled non-linear integro-differential equations that provide a non-perturbative resummation of select one-loop terms as can be seen by looking at a perturbative expansion of the solution to these equations.

We want to emphasize this point. By incorporating the full value of ϕ in the evolution equation for the mode functions we are incorporating back-reaction effects. If only the classical evolution of ϕ is used in the mode equations, we would have parametric resonant amplification since the effective frequencies for the mode functions are periodic due to the fact that the classical solution is periodic with constant amplitude. This leads to particle production that never shuts-off. However particle production leads to damping in the evolution of ϕ. Introducing this damped evolution in the mode equations leads to a behavior rather different from parametric resonance: as the evolution of ϕ is damped, the amplitude becomes smaller and particle production should diminish and eventually stop. This was found to be the situation in the case of the scalar field with self-interaction [13].

Clearly the integration of this set of equations will have to be done numerically. Because the self-interacting scalar case was already studied before [13], we will now concentrate on the interaction with the scalar fields in which thresholds are present. But before we do this we must understand the renormalization aspects of this system of equations.

The large-k behavior of the mode functions can be understood from a WKB analysis, the details of which had already been discussed in [8,13]. We find the following divergence structure doing this

$$\begin{aligned}
\int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{|U_k(t)|^2}{2W_k} &= \frac{\Lambda}{8\pi^2} - \frac{1}{8\pi^2}\left(m_{\Phi,b}^2 + \frac{\lambda_b}{2}\phi^2(t)\right) \ln\frac{\Lambda}{\kappa} + F_1(t,\kappa) \; , \\
\int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{|V_k(t)|^2}{2w_k} &= \frac{\Lambda}{8\pi^2} - \frac{1}{8\pi^2}\left(m_{\sigma,b}^2 + g_b\phi^2(t)\right) \ln\frac{\Lambda}{\kappa} + F_2(t,\kappa) \; ,
\end{aligned}$$

where Λ is an upper momentum cut-off and κ and arbitrary renormalization scale. The subscript b refers to bare quantities and the quantities $F_{1,2}(t,\kappa)$ are finite in the limit $\Lambda \to \infty$. We can now read the mass and coupling constant renormalizations

$$\begin{aligned}
m_{\Phi,R}^2 &= m_{\Phi,b}^2 + \frac{\Lambda}{8\pi^2}\left(\frac{\lambda_b}{2} + g_b\right) - \frac{1}{8\pi^2}\left(\frac{\lambda_b}{2}\, m_{\Phi,b}^2 + g_b \, m_{\sigma,b}^2\right) \ln\frac{\Lambda}{\kappa} \; , \\
\lambda_R &= \lambda_b - \frac{3}{4\pi^2}\left(\frac{{\lambda_b}^2}{4} + g_b^2\right) \ln\frac{\Lambda}{\kappa} \; .
\end{aligned}$$

We introduce a further, finite renormalization, by subtracting the functions $F_{1,2}(t=0,\kappa)$, absorbing this subtraction into a (finite) renormalization of the mass. After these renormalizations, we finally arrive at the renormalized set of evolution equations in terms of

renormalized quantities. We drop the subscript R for renormalized quantities to avoid cluttering the notation, but with the understanding that all quantities are renormalized at the scale κ:

$$\ddot{\phi}(t) + m_\Phi^2\, \phi(t) + \frac{\lambda}{6}\, \phi^3(t) + \frac{\lambda}{2}\, \phi(t) \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{|U_k(t)|^2 - 1}{2W_k} + g\, \phi(t) \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{|V_k(t)|^2 - 1}{2w_k}$$
$$+ \frac{1}{8\pi^2} \left(\frac{\lambda^2}{4} + g^2 \right) \phi(t) \left[\phi^2(t) - \phi^2(0) \right] \ln \frac{\Lambda}{\kappa} = 0 \; . \tag{3.5}$$

To this order, we can replace masses and couplings by their renormalized values in the equations for the mode functions (3.2, 3.3). We will now chose the renormalization scale to be $\kappa = m_\Phi$ so as to have only one scale in the problem, which makes the numerical evaluation easier.

The renormalized equation of motion (3.5) may be written without reference to the cutoff Λ which in the end must be taken to infinity, however, numerically the k-integrals must be calculated with an upper momentum cutoff. One must ensure that this numerical cutoff, to be identified with Λ in the evolution equation, be much larger than the masses and amplitudes of the field for the integrals to reach their asymptotics and the cutoff dependence to dissapear. We have been careful to make exhaustive checks that the final results were insensitive (to the working accuracy) to the cutoff which was typically chosen to be $\Lambda \approx 100\, |m_\Phi|$. This implies that the order of magnitude of the error is about $(\frac{m_\Phi}{\Lambda})^2 \simeq 10^{-4}$.

A provision must be made for the initial conditions on the mode functions solution of (3.2, 3.3) in the broken symmetry case. In this case the tree level squared mass is negative $m_\Phi^2 = -\mu^2$ and for initial values of the expectation value $\phi^2(0) < 2\mu^2/\lambda$ the initial configuration is below the classical "spinodal" and imaginary frequencies lead to the spinodal instabilities. We are not interested in this article in studying the time evolution of these instabilities but on the issues of non-linear relaxation. There are two ways to avoid the complex frequencies associated with these instabilities in the initial conditions: (i) one can choose an initial value $\phi^2(0) > 2\mu^2/\lambda$ or (ii) one can impose that the initial frequencies correspond to a positive mass term. This choice corresponds to preparing an initial gaussian density matrix of the modes with this given mass, under time evolution this packet spreads in function space and the time dependence of the width reflects the (quantum and thermal) fluctuations. In our numerical analysis we chose the later possibility with the positive mass squared given by the absolute value of the (negative) mass squared in the Lagrangian.

We introduce now the following dimensionless variables:

$$\eta(\tau) = \sqrt{\frac{\lambda}{6\, |m_\Phi^2|}} \phi(t) \; , \qquad q = \frac{k}{|m_\Phi|} \; , \qquad \tau = |m_\Phi|\, t \; ,$$
$$\bar{W}_q = \sqrt{q^2 + \frac{|M^2(0)|}{|m_\Phi^2|}} \; , \qquad \bar{w}_q = \sqrt{q^2 + \frac{m^2(0)}{|m_\Phi^2|}} \; .$$

In terms of which the evolution equation (3.5) becomes

$$\ddot{\eta}(\tau) + \eta(\tau) + \eta^3(\tau) + \frac{\lambda}{8\pi^2}\, \eta(\tau) \int^{\frac{\Lambda}{|m_\Phi|}} q^2 dq\, \frac{|U_q(\tau)|^2 - 1}{\bar{W}_q} + \frac{g}{4\pi^2}\, \eta(\tau) \int^{\frac{\Lambda}{|m_\Phi|}} q^2 dq\, \frac{|V_q(\tau)|^2 - 1}{\bar{w}_q}$$
$$+ \frac{\lambda}{8\pi^2} \left(\frac{3}{2} + \frac{6g^2}{\lambda^2} \right) \eta(\tau) \left[\eta^2(\tau) - \eta^2(0) \right] \ln \frac{\Lambda}{|m_\Phi|} = 0 \; , \tag{3.6}$$

where we chose the renormalization scale $\kappa = |m_\Phi|$.

1. **Particle Production**

As in any time dependent situation, the concept of particle is ambiguous and has to be specified with respect to some particular state. We choose that state as the equilibrium ensemble at the initial time $t = 0$. The initial condition on the mode functions for the fluctuations (3.2, 3.3) naturally determine the set of positive and negative energy states at this initial time. At this time the fluctuation operators may be expanded in this basis. The Fourier components of the fluctuation operators are thus written as

$$\chi_{\vec{k}}(0) = \frac{1}{\sqrt{2W_k}}\left[a_{\vec{k}}(0) - a^\dagger_{\vec{k}}(0)\right] \ ,$$
$$\sigma_{\vec{k}}(0) = \frac{1}{\sqrt{2w_k}}\left[b_{\vec{k}}(0) - b^\dagger_{\vec{k}}(0)\right] \ .$$

The number operators at any time t are

$$N_{\chi,k}(t) = \frac{\mathrm{Tr}\, a^\dagger_{\vec{k}}(0) a_{\vec{k}}(0)\rho(t)}{\mathrm{Tr}\,\rho(0)} = \frac{\mathrm{Tr}\, a^\dagger_{\vec{k}}(t) a_{\vec{k}}(t)\rho(0)}{\mathrm{Tr}\,\rho(0)} \ ,$$
$$a_{\vec{k}}(t) = U(t) a_{\vec{k}}(0) U^{-1}(t) \ ,$$
$$N_{\sigma,k}(t) = \frac{\mathrm{Tr}\, b^\dagger_{\vec{k}}(0) b_{\vec{k}}(0)\rho(t)}{\mathrm{Tr}\,\rho(0)} = \frac{\mathrm{Tr}\, b^\dagger_{\vec{k}}(t) b_{\vec{k}}(t)\rho(0)}{\mathrm{Tr}\,\rho(0)} \ ,$$
$$b_{\vec{k}}(t) = U(t) b_{\vec{k}}(0) U^{-1}(t) \ ,$$

with $U(t)$ the unitary time evolution operator. Following the arguments presented in reference [13] we find that the creation and annihilation operators at time t are related to those at the initial time $t = 0$ by a Bogoliubov transformation. In terms of the dimensionless variables introduced above we find

$$N_{\chi,q}(\tau) = (2\mathcal{F} - 1)\, N_{\chi,q}(0) + (\mathcal{F} - 1) \ ,$$
$$\mathcal{F} = \frac{1}{4}|U_q(\tau)|^2 \left[1 + \frac{|\dot{U}_q(\tau)|^2}{\bar{W}_q^2 |U_q(\tau)|^2}\right] + \frac{1}{2} \ ,$$
$$N_{\sigma,q}(\tau) = (2\mathcal{G} - 1)\, N_{\sigma,q}(0) + (\mathcal{G} - 1) \ ,$$
$$\mathcal{G} = \frac{1}{4}|V_q(\tau)|^2 \left[1 + \frac{|\dot{V}_q(\tau)|^2}{\bar{w}_q^2 |V_q(\tau)|^2}\right] + \frac{1}{2} \ ,$$

where $N_{\chi,\sigma}(0)$ are the occupation numbers at the initial time $t = 0$ and in the case given by the Bose-Einstein distribution functions, derivatives are with respect to the dimensionless variable τ. Since in this section we will be working at zero temperature, only the last term will contribute to particle production. This term is recognized as the "induced" contribution. It can be seen from the initial conditions on the wave functions that the induced contribution vanishes at the initial time.

We will compute numerically the number of particles produced in a correlation volume

$$\mathcal{N}^b(\tau) = \int d^3k N_k(t)/|m_\Phi|^3 \ . \tag{3.7}$$

B. Inflaton Coupled to Fermions Only

The fermionic Green's functions are constructed from the solutions to the homogeneous Dirac equation in the presence of the background field. The main ingredients and treatment is similar to that of fermions in presence of an electric field studied by Kluger et. al. [23]. Writing the independent solutions as

$$\mathcal{U}^{(1,2)}(\vec{x},t) = e^{i\vec{k}\cdot\vec{x}}U^{(1,2)}_k(t) \; ,$$
$$\mathcal{V}^{(1,2)}(\vec{x},t) = e^{-i\vec{k}\cdot\vec{x}}V^{(1,2)}_k(t) \; ,$$

the mode functions obey

$$\left[i\gamma_0\frac{d}{dt} - \vec{\gamma}\cdot\vec{k} - m_\psi(t)\right]U^{(1,2)}_k(t) = 0 \; ,$$
$$\left[i\gamma_0\frac{d}{dt} + \vec{\gamma}\cdot\vec{k} - m_\psi(t)\right]V^{(1,2)}_k(t) = 0 \; .$$

It is convenient to write the spinors as

$$U^{(1,2)}_k(t) = \left[i\gamma_0\frac{d}{dt} - \vec{\gamma}\cdot\vec{k} + m_\psi(t)\right]f_k(t)u^{(1,2)} \; ,$$
$$V^{(1,2)}_k(t) = \left[i\gamma_0\frac{d}{dt} + \vec{\gamma}\cdot\vec{k} + m_\psi(t)\right]g_k(t)v^{(1,2)} \; ,$$

with $u^{(1,2)}$, $v^{(1,2)}$ the spinor eigenstates of γ_0 with eigenvalues $+1$, -1 respectively. The functions $f_k(t)$, $g_k(t)$ obey the second order equations

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2_\psi(t) - i\dot{m}_\psi(t)\right]f_k(t) = 0 \; ,$$
$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2_\psi(t) + i\dot{m}_\psi(t)\right]g_k(t) = 0 \; .$$

We now need to append initial conditions. We will consider the situation in which the system was in equilibrium at time $t \leq 0$ with the expectation value of scalar field being $\phi(0)$ and $\dot{\phi}(0) = 0$. Thus the fermion mass is constant and given by $m_\psi(0)$. We can now impose the condition that the modes $f_k(t)$, $g_k(t)$ describe positive and negative frequency solutions for $t \leq 0$ and, normalizing the spinor solutions to the Dirac equation to unity, we impose the following initial conditions

$$f_k(t<0) = \frac{e^{-ik_0t}}{\sqrt{2k_0(k_0+m_\psi(0))}} \; ,$$
$$g_k(t<0) = \frac{e^{ik_0t}}{\sqrt{2k_0(k_0+m_\psi(0))}} \; ,$$
$$k_0 = \sqrt{\vec{k}^2 + m^2_\psi(0)} \; .$$

Equations (3.8) with these boundary conditions imply that

$$g_k(t) = f^*_k(t) \ . \tag{3.8}$$

The necessary ingredients for the zero temperature fermionic Green's functions are the following

$$\begin{aligned} S^>_k(t,t') &= -i \sum_{\alpha=1,2} U^\alpha_{\vec{k}}(t) \bar{U}^\alpha_{\vec{k}}(t') \ , \\ S^<_k(t,t') &= i \sum_{\alpha=1,2} V^\alpha_{-\vec{k}}(t) \bar{V}^\alpha_{-\vec{k}}(t') \ . \end{aligned}$$

In the standard Dirac representation for the γ matrices, we find

$$\begin{aligned} S^>_k(t,t') &= -i f_k(t) f^*_k(t') \left[\mathcal{W}_k(t)\gamma_0 - \vec{\gamma}\cdot\vec{k} + m_\psi(t)\right] \left(\frac{1+\gamma_0}{2}\right) \left[\mathcal{W}^*_k(t')\gamma_0 - \vec{\gamma}\cdot\vec{k} + m_\psi(t')\right] \ , \\ S^<_k(t,t') &= -i f^*_k(t) f_k(t') \left[\mathcal{W}^*_k(t)\gamma_0 - \vec{\gamma}\cdot\vec{k} - m_\psi(t)\right] \left(\frac{1-\gamma_0}{2}\right) \left[\mathcal{W}_k(t')\gamma_0 - \vec{\gamma}\cdot\vec{k} - m_\psi(t')\right] \ , \\ \mathcal{W}_k(t) &= i\frac{\dot{f}_k(t)}{f_k(t)} \ . \end{aligned}$$

With these ingredients the non-equilibrium fermionic Green's functions can be constructed as in section II. It is an important check that the equality $\mathrm{Tr}\, S^{++}_k(t,t) = \mathrm{Tr}\, S^{--}_k(t,t) = \mathrm{Tr}\, S^>_k(t,t) = \mathrm{Tr}\, S^<_k(t,t)$ is fulfilled where the trace is over Dirac indices. It is also an important property that

$$\left(-i\dot{f}^*_k(t) + m_\psi(t) f^*_k(t)\right)\left(i\dot{f}_k(t) + m_\psi(t) f_k(t)\right) + k^2 f^*_k(t) f_k(t) = 1 \ . \tag{3.9}$$

This is a consequence of the conservation of probability and may be checked explicitly.

The evolution equation becomes

$$\begin{aligned} &\ddot{\phi}(t) + m^2_\Phi\, \phi(t) + \frac{\lambda}{6}\,\phi^3(t) + \frac{\lambda}{2}\,\phi(t) < \chi^2(\vec{x},t) > -y\, \mathrm{Tr}\, S^<(\vec{x},t;\vec{x},t) = 0 \ , \\ &\mathrm{Tr}\, S^<(\vec{x},t;\vec{x},t) = 2\int \frac{d^3k}{(2\pi)^3}\left[1 - 2k^2 |f_k(t)|^2\right] \ . \end{aligned} \tag{3.10}$$

The integral in (3.10) is divergent. The divergence structure may be understood in two different ways: carrying out a WKB expansion of the modes, just as was done in the bosonic case (this is a tedious and lengthy exercise), or alternatively calculating the closed loop in the presence of the background field. We carried out both methods using an upper momentum cutoff in the integrals and found

$$2y\int \frac{d^3k}{(2\pi)^3}\left[1 - 2\,k^2 |f_k(t)|^2\right] = -\frac{y}{2\pi^2}\,\Lambda^2\; m_\psi(t) + \frac{y}{4\pi^2}[\ddot{m}_\psi(t) + 2\,m^3_\psi(t)]\ln\frac{\Lambda}{\kappa} + \text{ finite} \ , \tag{3.11}$$

where κ is an arbitrary renormalization scale which will be chosen again to be $\kappa = m_\Phi$ for numerical convenience. We will concentrate on the "chiral limit" in which the *bare* mass

term for the fermions vanishes and $m_\psi(t) = y\,\phi(t)$. In this limit the discrete symmetry $\phi \to -\phi$ is explicit. The divergent terms in (3.11) are then identified as: mass renormalization (of ϕ) (first term), wave function renormalization (second term) and coupling constant renormalization (third term). From now on, all quantities will be renormalized, and we drop the subscript R to avoid cluttering. For simplicity in the numerical analysis and in order to isolate the fluctuations from the fermionic contribution from those of the scalar sector, we now neglect the contribution from the scalar loop. The scalar fluctuations had already been analyzed in the previous section and in [13]. The final renormalized equation of motion is now

$$\ddot{\phi}(t) + m_\Phi^2\,\phi(t) + \frac{\lambda}{6}\phi^3(t) - 2y\int \frac{d^3k}{(2\pi)^3}\left[1 - 2k^2|f_k(t)|^2\right] - \left\{ -\frac{y^2}{2\pi^2}\Lambda^2\,\phi(t) \right.$$
$$\left. + \frac{y^2}{4\pi^2}\left[\ddot{\phi}(t) + 2y^2\,\phi^3(t)\right]\ln\frac{\Lambda}{m_\Phi} \right\} = 0\;.$$

With the purpose of a numerical analysis of this equations, it proves convenient to introduce the following dimensionless of variables

$$\eta = y\phi/m_\Phi\;,\quad \tau = m_\Phi t\;,\quad q = k/m_\Phi\;,\quad g' = \lambda_R/6y^2\;,\quad g = y^2/\pi^2\;,$$
$$u_q(\tau) = f_k(t)\sqrt{\omega_k^0(\omega_k^0 + y^2\phi^2(0))}\;,$$

in terms of which the equations of motion are

$$\frac{d^2\eta}{d\tau^2} + \eta + g'\eta^3 - g\Sigma(\tau) = 0\;,$$
$$\left[\frac{d^2}{d\tau^2} + q^2 + \eta^2 - i\dot{\eta}\right]u_q(\tau) = 0\;,\qquad u_q(0) = \frac{1}{\sqrt{2}}\;,\qquad \dot{u}_q(0) = -i\frac{\sqrt{q^2+\eta^2(0)}}{\sqrt{2}}\;,$$
$$\Sigma(\tau) = \int_0^{\Lambda/m_\Phi} q^2\,dq\left[1 - \frac{q^2|u_q(\tau)|^2}{\sqrt{q^2+\eta^2(0)}\left(\sqrt{q^2+\eta^2(0)}+\eta(0)\right)}\right]$$
$$-\frac{1}{2}\left(\frac{\Lambda}{m_{R,\Phi}}\right)^2\eta + \frac{1}{4}\left(\ddot{\eta} + 2\,\eta^3\right)\ln\frac{\Lambda}{m_{R,\Phi}}\;. \tag{3.12}$$

Although the equations are finite when the ultraviolet cutoff is taken to infinity and can be written without the introduction of the cutoff by subtracting the integral with a *lower* limit cutoff given by the renormalization scale, numerically these integrals will have to be done by introducing an upper momentum cutoff anyways. Therefore we keep this UV cutoff in the equations.

1. **Particle Production**

The study of particle production is similar to the scalar case, with the only complication being the spinorial structure of the fermionic fields. At the initial time $t = 0$ the quantized fermion operator is

$$\psi(\vec{x},0)=\int\frac{d^3k}{(2\pi)^3}\sum_{\alpha=1,2}\left[b_k^{(\alpha)}(0)U_k^{(\alpha)}(0)+d_{-k}^{(\alpha)\dagger}(0)V_{-k}^{(\alpha)}(0)\right]e^{i\vec{k}\cdot\vec{x}}\;, \tag{3.13}$$

in terms of the mode functions determined above. The number of fermions and antifermions are *defined* as

$$\langle N^f(t)\rangle=\int\frac{d^3k}{(2\pi)^3}\sum_\alpha\frac{\mathrm{Tr}\left[\rho(t)b_k^{(\alpha)\dagger}(0)b_k^{(\alpha)}(0)\right]}{\mathrm{Tr}\,\rho(0)}=\int\frac{d^3k}{(2\pi)^3}\sum_\alpha\frac{\mathrm{Tr}\left[\rho(0)b_k^{(\alpha)\dagger}(t)b_k^{(\alpha)}(t)\right]}{\mathrm{Tr}\,\rho(0)}\;,$$
$$\langle N^{\bar{f}}(t)\rangle=\int\frac{d^3k}{(2\pi)^3}\sum_\alpha\frac{\mathrm{Tr}\left[\rho(t)d_k^{(\alpha)\dagger}(0)d_k^{(\alpha)}(0)\right]}{\mathrm{Tr}\,\rho(0)}=\int\frac{d^3k}{(2\pi)^3}\sum_\alpha\frac{\mathrm{Tr}\left[\rho(0)d_k^{(\alpha)\dagger}(t)d_k^{(\alpha)}(t)\right]}{\mathrm{Tr}\,\rho(0)}\;.$$

The time dependent coefficients are obtained by projecting the time dependent spinor solutions onto the positive and negative energy solutions at $t=0$, and are related to the coefficients at $t=0$ via a Bogoliubov transformation

$$b_k^{(\alpha)}(t)=\sum_\beta\left[\mathcal{F}^{(\alpha)}_{(\beta)k,+}(t)b_k^{(\beta)}+\mathcal{F}^{(\alpha)}_{(\beta)k,-}(t)d_{-k}^{(\beta)\dagger}\right]\;,$$
$$d_{-k}^{(\alpha)\dagger}(t)=\sum_\beta\left[\mathcal{H}^{(\alpha)}_{(\beta)k,+}(t)b_k^{(\beta)}+\mathcal{H}^{(\alpha)}_{(\beta)k,-}(t)d_{-k}^{(\beta)\dagger}\right]\;.$$

The identities

$$\sum_\beta\mid\mathcal{F}^{(\alpha)}_{(\beta)k,+}(t)\mid^2+\mid\mathcal{F}^{(\alpha)}_{(\beta)k,-}(t)\mid^2=1\;,$$
$$\sum_\beta\mid\mathcal{H}^{(\alpha)}_{(\beta)k,+}(t)\mid^2+\mid\mathcal{H}^{(\alpha)}_{(\beta)k,-}(t)\mid^2=1$$

ensure that the transformations preserve the anticommutation relations as they must. It is also a matter of algebra using the equations of motion for the mode functions to prove that the number of fermions minus antifermions is conserved for each $\vec{k}$ mode. In what follows, we only consider the number of fermions produced since fermions and antifermions are created in pairs.

With some algebra, the Bogoliubov coefficients can be expressed as the function of $f_k(t)$ and $f_k^*(t)$. In terms of the dimensionless variables defined above we obtain the number of fermions within a correlation volume $\mathcal{N}^f(\tau)=N^f(\tau)/|m_\Phi|^3$

$$\begin{aligned}\mathcal{N}^f(\tau)&=\frac{1}{2\pi^2}\int q^2\,dq\mathcal{N}^f_q(\tau)\\&=\frac{1}{4\pi^2}\int dq\left[\frac{q^2}{\sqrt{q^2+\eta^2(0)}\left(\sqrt{q^2+\eta^2(0)}+\eta(0)\right)}\right]^2\\&\quad\times\left[-i\frac{\partial}{\partial\tau}+\eta(\tau)-\sqrt{q^2+\eta^2(0)}-\eta(0)\right]u_q^*(\tau)\\&\quad\times\left[i\frac{\partial}{\partial\tau}+\eta(\tau)-\sqrt{q^2+\eta^2(0)}-\eta(0)\right]u_q(\tau)\;.\end{aligned}\tag{3.14}$$

IV. NUMERICAL ANALYSIS

The numerical analysis was carried out with a fourth order Runge-Kutta method for the differential equations and a 5-point Bode rule integrator for the k-integrals. The typical step size in time was $1 - 2.10^{-3}$, and the typical step size in k was 10^{-3}. The cutoff $\Lambda/|m_\Phi|$ was varied between 75 and 200; we found no sensitivity to the cutoff in the range of parameters that we tested (see below). The code was very stable within the range of parameters tested.

A. Inflaton Coupled to Scalars: Unbroken Symmetry Case

Since the contribution of the one-loop quantum fluctuations of the inflaton Φ have already been studied previously [13] and we want to study the contribution of the lighter field σ, we will neglect the contribution from the self-interaction in the evolution equation. Thus we study the following equations obtained from equations (3.3, 3.6) in terms of dimensionless variables

$$\ddot{\eta}(\tau) + \eta(\tau) + \eta^3(\tau) + \frac{g}{4\pi^2}\,\eta(\tau)\int^{\frac{\Lambda}{|m_\Phi|}} q^2 dq\; \frac{|V_q(\tau)|^2 - 1}{\bar{w}_q}$$

$$+\frac{\lambda}{8\pi^2}\,\frac{6g^2}{\lambda^2}\,\eta(\tau)\left[\eta^2(\tau) - \eta^2(0)\right]\ln\frac{\Lambda}{|m_\Phi|} = 0\;,$$

$$\left[\frac{d^2}{d\tau^2} + \vec{q}^{\,2} + \frac{m_\sigma^2}{|m_\Phi|^2} + \frac{6g}{\lambda}\eta(\tau)\right]V_q(\tau) = 0\;,$$

$$V_k(0) = 1\;,\quad \dot{V}_q(0) = -i\,\bar{w}_q = -i\sqrt{\vec{q}^{\,2} + \frac{m_\sigma^2}{|m_\Phi|^2} + \frac{6g}{\lambda}\eta^2(0)}\;.$$

Notice that the factor (3/2) multiplying the logarithm in eq. (3.6) and missing from (4.1) arises from the renormalization of the Φ-scalar loop that is not taken into account in (4.1).

Figures (1.a-c) show the unbroken symmetry case with the quantum fluctuations from the scalar loop of the σ particles for the values $y = 0$; $\lambda/8\pi^2 = 0.2$; $g = \lambda$; $m_\sigma = 0.2\,m_\Phi$; $\eta(0) = 1.0$; $\dot{\eta}(0) = 0$. Figure (1.a) shows $\eta(\tau)$ vs τ, figure (1.b) shows $\mathcal{N}_\sigma(\tau)$ vs τ and figure (1.c) shows $\mathcal{N}_{q,\sigma}$ vs q for $\tau = 120$, similar graphs were obtained with snapshots at different (earlier) times.

Figure (1.a) shows a very rapid, non-exponential damping within few oscillations of the expectation value and a saturation effect when the amplitude of the oscillation is rather small (about 0.1 in this case), the amplitude remains almost constant at the latest times tested. Figure (1.a) and figure (1.b) clearly show that the time scale for dissipation (from figure (1.a) is that for which the particle production mechanism is more efficient (figure (1.b)). Notice that the total number of particles produced rises on the same time scale as that of damping in figure (1.a) and eventually when the expectation value oscillates with (almost) constant amplitude the average number of particles produced remains constant. These figures clearly show that damping is a consequence of particle production. At times larger than about $40\,m_\Phi^{-1}$ (for the initial values and couplings chosen) there is no appreciable damping. The amplitude is rather small and particle production has practically shut off. If we had used the *classical* evolution of the expectation value in the mode equations, particle

production would not shut off (parametric resonant amplification), and thus we clearly see the dramatic effects of the inclusion of the back reaction.

In this unbroken symmetry case linear relaxation predicts a slow t^{-3} power law decay to an asymptotic finite amplitude because one particle decay is kinematically forbidden: the self energy contribution is a two-loop effect with one Φ and two σ particle cut and kinematically there is a one-particle pole below the three particle threshold. A slow power law linear relaxation asymptotically cannot be ruled out numerically because we have not continued the integration for longer times but clearly asymptotically the numerical result is compatible with linear relaxation. Figure (1.c) shows the distribution of particles created at the latest time $\tau = 120$ as a function of wave vector. Similar graphs were obtained with snapshots at different earlier times. The distribution is clearly non-thermal and skewed towards small momentum (in units of m_Φ). These figures are qualitatively similar to those obtained in the self-interacting case in [13]. The asymptotic behavior is that of undamped oscillations of small amplitude. This is compatible with the result from the linear relaxation analysis because there is a one-particle pole below the three particle threshold resulting in undamped oscillations at large times. Linear relaxation predicts qualitatively the same results for the self-interacting scalar case and this case in the unbroken phase. The numerical results are consistent with this prediction.

However, for large amplitudes non-linear relaxation via particle production is very effective and dramatically different from linear relaxation.

B. Inflaton Coupled to Scalars: Broken Symmetry Case

In this case, we take $m^2_\Phi = -|m^2_\Phi|$.

As in the unbroken symmetry case, we only study the effect of the lighter σ fluctuations. In the broken symmetry case, linear relaxation predicts an open decay channel for the inflaton, resulting in exponential relaxation for a long time and eventually relaxation with a power law. Therefore we should not expect a constant amplitude asymptotically but an amplitude that eventually should relax to zero. In this case the renormalized equations for evolution are

$$\ddot{\eta}(\tau) - \eta(\tau) + \eta^3(\tau) + \frac{g}{4\pi^2}\,\eta(\tau)\int^{\frac{\Lambda}{|m_\Phi|}} q^2 dq\, \frac{|V_q(\tau)|^2 - 1}{\bar{w}_q}$$
$$+\frac{\lambda}{8\pi^2}\,\frac{6g^2}{\lambda^2}\,\eta(\tau)\left[\eta^2(\tau) - \eta^2(0)\right]\ln\frac{\Lambda}{|m_\Phi|} = 0\;,$$
$$\left[\frac{d^2}{d\tau^2} + \vec{q}^2 + \frac{m^2_\sigma}{|m_\Phi|^2} + \frac{6g}{\lambda}\eta(\tau)\right] V_q(\tau) = 0\;,$$
$$V_k(0) = 1\;,\quad \dot{V}_q(0) = -i\,\bar{w}_q = -i\sqrt{\vec{q}^2 + \frac{m^2_\sigma}{|m_\Phi|^2} + \frac{6g}{\lambda}\eta^2(0)}\;.$$

Figure (2.a-c) show $\eta(\tau)$ vs τ, $\mathcal{N}_\sigma(\tau)$ vs τ and $\mathcal{N}_{q,\sigma}(\tau = 200)$ vs q respectively, for $\lambda/8\pi^2 = 0.2$; $g/\lambda = 0.05$; $m_\sigma = 0.2\,|m_\Phi|$; $\eta(0) = 0.6$; $\dot{\eta}(0) = 0$. Notice that the mass for the linearized perturbations of the Φ field at the broken symmetry ground state is $\sqrt{2}\,|m_\Phi| > 2m_\sigma$. Therefore, for the values used in the numerical analysis, the two-particle decay channel

is open for linear relaxation. For these values of the parameters, linear relaxation predicts exponential decay with a time scale $\tau_{rel} \approx 300$ (in the units used). Figure (2.a) shows very rapid non-exponential damping on time scales about *six times shorter* than that predicted by linear relaxation. The expectation value reaches very rapidly a small amplitude regime, once this happens its amplitude relaxes very slowly. Within our computing time limitations we could not confirm that there is exponential relaxation in the small amplitude regime (for $\tau > 100$) but clearly there is a striking difference with the unbroken symmetry case. The influence of open channels is evident, however in the non-linear regime relaxation is clearly *not* exponential but extremely fast. Although we cannot confirm the exponential (or power law) relaxation numerically in the small amplitude regime, the amplitude at long times seems to relax to the expected value, shifted slightly from the minimum of the tree level potential at $\eta = 1$. This is as expected from the fact that there are quantum fluctuations. Figure (2.b) shows that particle production occurs during the time scale for which dissipation is most effective, giving direct proof that dissipation is a consequence of particle production. Asymptotically, when the amplitude of the expectation value is small, particle production shuts off. We point out again that this is a consequence of the back-reaction in the evolution equations. Without this back-reaction, as argued above, particle production would continue without indefinitely. Figure (2.c) shows that the distribution of produced particles is very far from thermal and concentrated at low momentum modes $k \leq |m_\Phi|$. This distribution is qualitatively similar to that in the unbroken symmetry case, and points out that the excited state obtained asymptotically is far from thermal.

C. Inflaton Coupled to Fermions Only: Unbroken Symmetry Case

Here we treat the case $y \neq 0; \quad g = 0$.

The renormalized evolution and particle production equations in this case are given by eq. (3.12) and eq. (3.14), respectively, in terms of dimensionless quantities. Figures (3.a-c) show $\eta(\tau)$ vs. τ, $\mathcal{N}^f(\tau)$ vs τ and $\mathcal{N}^f_q(\tau = 200)$ respectively for the values of the parameters $m_\psi = 0; \quad y^2/\pi^2 = 0.5; \quad \lambda/6y^2 = 1.0; \quad \eta(0) = 0.6; \quad \dot{\eta}(0) = 0$. One observes from these figures that after a rather brief period of initial damping of just a few oscillations of the scalar field, the dissipative mechanism shuts-off. Figure (3.b) shows that during this time scale fermion-antifermion pairs are being produced but then the number of produced particles saturates and oscillates with a small amplitude. Figure (3.c) shows that the distribution of fermions produced is peaked at very low momentum ($k \leq m_\Phi$) and with a maximum value of 2, which is the total number of degrees of freedom per k-wave vector. These numerical results expose the physics of Pauli blocking very clearly; the available low momentum modes are occupied and no more fermion-antifermion pairs can be produced. Pauli blocking shuts off particle production and dissipation very early on. We have obtained snapshots of the particle number as a function of momentum for different times, and they all present the same picture.

This result is markedly different from the prediction of linear relaxation. At zero temperature, eq. (2.48) for linear relaxation predicts exponential damping with a (dimensionless) time scale $\tau_{rel} \approx 10.6$ for the values of the parameters chosen above. The difference between the non-linear evolution and that predicted by linear relaxation is explained by the Pauli blocking phenomenon. Very early in the evolution, the low momentum available fermionic

states were filled with produced fermions. Once these states have filled, damping and particle production shuts off. This Pauli blocking effect is explicit in eq. (2.48) but there it appears from finite temperature effects. In the non-linear relaxation case, we began at zero temperature, but an excited state quickly ensues because of particle production.

This analysis reveals that for large amplitudes of the scalar field, fermions will be rather ineffective in dissipation and damping because of Pauli blocking *even at zero temperature*. The time scales for dissipation obtained from the fermion self-energies, which apply to the case of linear relaxation are completely unrelated to the time scales for non-linear dissipative processes, even asymptotically, because the fermionic states are Pauli blocked.

V. CONCLUSIONS AND IMPLICATIONS

The reheating and thermalization processes in inflationary universe models occur very far from equilibrium and must be studied in their full complexity, eventually numerically. The methods developed within real-time non-equilibrium field theory allow a consistent treatment that we used to obtain the equations of evolution for the expectation value of the inflaton field. These equations are non-perturbative and take into account the back reaction effect of both the self couplings of the inflaton, as well as the couplings to lighter scalars and fermions.

We have examined these evolution equations, both in the linear regime, in which the amplitude of the field is small and in the opposite extreme, the non-linear regime, which is intrinsically non-perturbative.

In the linear regime, we have seen that if the inflaton mass is above the two particle threshold, there is some damping behavior for a period of time, and indeed, this damping is governed by the total width of the inflaton. However, at late times the behavior is dominated by *power law* decay rather than exponential damping. Furthermore, we have noted the important distinction between the time scale for the relaxation of the expectation value of the inflaton as opposed to thermalization of the produced particles. It is clear that as treated in this work, relaxation can occur well before thermalization, i.e. interactions that drive the particle distributions towards a thermal one, has a chance to become relevant.

The more interesting case, and the one which would most likely be relevant to the reheating problem in models such as chaotic inflation [5], is that of when the inflaton is in the non-linear regime. Unlike the elementary approach to reheating, in which the "decay" of the coherent inflaton oscillations occured a time $t_{\rm decay} \sim \Gamma^{-1}$, where Γ is the inflaton decay width, after the end of the slow-roll regime, particle production in the non-linear regime begins at very early times and then shuts off. Indeed, by the time the inflaton expectation value reaches its asymptotic state of oscillations around its minimum, the period of particle production is essentially over. This will be the stage of thermalization via collisions, however we find that the spectrum of produced particles is extremely non-thermal, and collisional relaxation towards an equilibrium thermal state may take a long time.

An inflationary model that cannot reheat the universe to at least nucleosynthesis temperatures is wrong. Thus it is clearly important to be able to compute the reheating temperature in such a model. What our analysis here shows is that this computation is of necessity much more complicated than has previously been thought but perhaps more importantly that there are different time scales.

The first stage which occurs rather fast is that of particle production and damping of oscillations as a result of induced amplification. The second stage, that begins typically when the amplitude of the inflaton is rather small and it oscillates with almost constant amplitude at the bottom of the potential is that of one-particle decay and of thermalization via collisional relaxation.

The thermalization time will have to be studied setting up a (quantum) Boltzmann equation using the distributions of produced particles at the end of the induced amplification stage as input for the Boltzmann evolution. Since these initial distributions are very far from equilibrium, it may take many collisions to relax to a thermal equilibrium state. If the couplings are very small (and this will clearly depend on the models) the thermalization time may be very large.

Following the particle numbers to the point at which they become thermal would then allow us to pick off the final reheating temperature. In the weak coupling regime, as mentioned above, the thermalization rate may be smaller than the expansion rate, allowing for significant redshifting of the total energy density, leading to a low reheat temperature, perhaps of the order of the inflaton mass or less.

The particular case of large thermalization rates compared to the expansion rate, allows for a "quick" estimate of the reheating temperature. In this case we can assume that thermalization occurs without red-shifting of the energy which is then conserved during the thermalization time. By taking the energy density at the beginning of this stage to be αT^4 with α the Stephan Boltzmann accounting for the particle statistics one can then obtain an estimate of the reheating temperature, but the justification for this should ultimately arise from a deeper understanding of the time scales involved.

Our main point in all of this, though, is that a more detailed calculation will have to be done in order to extract the reheating temperature in a more reliable way. However, what we have done is to clarify what the relevant particle production mechanisms are in these theories and to provide a consistent and implementable framework of calculation. We are now extending these studies to FRW cosmologies.

We can compare our work to other recent work in this subject. Brandenberger et al [12] have performed a calculation of particle production for an oscillating inflaton field. However, they have used the inflaton field as a background which does *not* respond to the produced particles. In other words they neglect the back reaction, which as we have seen, allows for the production process to shut off, and determines the time scales for particle production. Kofman, Linde and Starobinsky [11] have also performed such a calculation, and where we have common values of the parameters, we agree. However, they do not follow the produced particles through the thermalization period, so that we are somewhat uncertain about how they draw their conclusions about the final reheating temperature. We emphasized in this study that thermalization and particle production are fundamentally different processes and in the non-linear regime likely to occur on widely different time scales.

One area of current interest in which this work may have some implications is that of the so-called "post-modern Polonyi problem" concerning flat directions for some of the moduli fields in string theories [17]. These are fields with pertubatively flat directions whose degeneracy is lifted by non-perturbative effects. Their masses then become of order the weak scale and their couplings to normal matter are gravitational. These properties allow the energy density in these fields to dominate that of the radiation in the universe until

times well after nucleosynthesis.

This may not necessarily be true given our analysis. Since particle production occurs not just during oscillations around the minimum now but throughout the evolution of the the moduli field, it is not impossible that some of this energy density could be transferred to lighter particles, prior to nucleosynthesis, perhaps averting this potential catastrophe. We are currently looking into this possibility.

To reiterate then: induced amplification during the evolution of the inflaton field allows for a very different mechanism for particle production at the end of reheating than has been used in the past for inflationary models. In the non-linear regime, this is by far the most important such process, and will have significant consequences for inflationary universe models.

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APPENDIX A: A PEDAGOGICAL EXERCISE

In this appendix we show explicitly how to implement the tadpole method for the case of the inflaton coupled to a lighter scalar field via a simple trilinear coupling. We will not worry about renormalization issues here, though they were, of course, dealt with in the text. The formalism for non-equilibrium quantum field theory has already been described several times in the literature [26]. A path integral representation involves an integration along a path in the complex time plane with forward, backward branches and if the initial density matrix was that of an equilibrium system at an (initial) temperature T also an imaginary time branch. For real time correlation functions the imaginary time branch does not contribute and only determines the boundary conditions on the Green's functions.

We use the tadpole method [8,13,25] to study the time evolution of the expectation value of the inflaton.

$$\phi(\vec{x},t) \equiv \frac{\mathrm{Tr}\,[\Phi^+(\vec{x})\rho(t)]}{\mathrm{Tr}\,\rho(0)} = \frac{\mathrm{Tr}\,[\Phi^-(\vec{x})\rho(t)]}{\mathrm{Tr}\,\rho(0)}\ , \tag{A1}$$

where $\Phi^{\pm}$ are the fields defined on the forward and backward branches respectively. We set

$$\Phi^{\pm}(\vec{x},t) = \phi(\vec{x},t) + \chi^{\pm}(\vec{x},t)\ , \tag{A2}$$

where $\chi^{\pm}$ are field fluctuation operators defined along the respective branches.

The effective evolution equation for the background field $\phi(\vec{x},t)$ follows from the condition.

$$< \chi^{\pm}(\vec{x},t) >= 0 \; . \tag{A3}$$

Treating the *linear* and non-linear terms in $\chi^{\pm}$ as interactions and imposing the condition (A3) consistently in a perturbative or loop expansion, one obtains expressions of the form (here we quote the equation obtained from $< \chi^{+}(\vec{x},t) >= 0$)

$$\int d\vec{x}' dt' \left[< \chi^{+}(\vec{x},t)\chi^{+}(\vec{x}',t') > \mathcal{O}^{++}(\vec{x}',t') + < \chi^{+}(\vec{x},t)\chi^{-}(\vec{x}',t') > \mathcal{O}^{+-}(\vec{x}',t') \right] = 0 \tag{A4}$$

and similarly for $< \chi^{-}(\vec{x},t) >= 0$. The $\mathcal{O}^{\pm\pm}$ are in general integro-differential operators acting on the background field.

Because the Green's functions $< \chi^{+}(\vec{x},t)\chi^{+}(\vec{x}',t') >$, etc. are all independent one obtains the equations of motion in the form

$$\mathcal{O}^{++}(\vec{x}',t') = 0 \; , \quad \mathcal{O}^{+-}(\vec{x}',t') = 0 \; , \quad \mathcal{O}^{-+}(\vec{x}',t') = 0 \; , \quad \mathcal{O}^{--}(\vec{x}',t') = 0 \; . \tag{A5}$$

It is a consequence of the properties of the non-equilibrium Green's function (see below) and ultimately a consequence of unitarity that all the integro-differential operators $\mathcal{O}$ are the same.

Consider the Lagrangian density

$$\mathcal{L}(\Phi,\sigma) = \mathcal{L}_0(\Phi) + \mathcal{L}_0(\sigma) + g(t) \; \Phi \; \sigma^2 \; , \tag{A6}$$

with the $\mathcal{L}_0$ being the free field Lagrangian density (with respective mass terms) and have allowed the coupling to depend on time. The non-equilibrium path integral requires $\mathcal{L}(\Phi^{+},\sigma^{+}) - \mathcal{L}(\Phi^{-},\sigma^{-})$. In the tadpole method we write $\Phi^{\pm}(\vec{x},t) = \chi^{\pm}(\vec{x},t) + \phi(t)$ and identify $\phi(t)$ as the (non-equilibrium) expectation value of the field Φ . This identification then requires that $< \chi(\vec{x},t) >= 0$ where the expectation value is in the non-equilibrium density matrix with the path integral representation along the contour in complex time. After this shift, the action reads:

$$\begin{aligned} L = \int d^3x dt \Big\{ & \mathcal{L}_0(\chi^{+}) + \mathcal{L}_0(\sigma^{+}) + \chi^{+} \left[-\ddot{\phi} - m_{\Phi}^2 \phi \right] + g(t)\phi(t)(\sigma^{+})^2 + g(t)\chi^{+}(\sigma^{+})^2 \\ & - \left(\chi^{+}, \sigma^{+} \to \chi^{-}, \sigma^{-} \right) \Big\} \; . \end{aligned}$$

The linear term in $\chi^{\pm}$ is included as a perturbation. The first contribution to the equation of motion is obtained from this linear term, from the condition $< \chi^{+}(\vec{x},t) >= 0$ one obtains to this order

$$\int dt' \left\{ < \chi^{+}(t)\chi^{+}(t') > (i) \left[-\ddot{\phi} - m_{\Phi}^2 \phi \right] - < \chi^{+}(t)\chi^{-}(t') > (i) \left[-\ddot{\phi} - m_{\Phi}^2 \phi \right] \right\} = 0 \; , \tag{A7}$$

where we have suppressed the spatial arguments. Because the correlation functions $< \chi^{+}(t)\chi^{+}(t') >$, $< \chi^{+}(t)\chi^{-}(t') >$ are independent, one obtains the tree level equations of motion. It is straightforward to see that the same is obtained by imposing $< \chi^{-}(\vec{x},t) >= 0$. In an amplitude expansion (an expansion in powers of $\phi(t)$) the one loop correction to the equation of motion is obtained by expanding (the exponential of) $g\phi(t)(\sigma^{+})^2 + g\chi^{+}(\sigma^{+})^2 - (+ \to -)$ to first and second order. The first order gives a tadpole contribution, the second order needs one vertex with χ, the other with ϕ. One obtains

$$
\begin{aligned}
&\int dt' < \chi^+(t)\chi^+(t') > \Big\{(i)\left(-\ddot{\phi} - m_\Phi^2\phi\right) + (ig(t')) < (\sigma^+(t'))^2 > + \\
&\int dt''(ig(t''))^2 \left[< (\sigma^+(t'))^2(\sigma^+(t''))^2 > - < (\sigma^+(t'))^2(\sigma^-(t''))^2 >\right]\phi(t'')\Big\} - \\
&\int dt' < \chi^+(t)\chi^-(t') > \Big\{(i)\left(-\ddot{\phi} - m_\Phi^2\phi\right) + (ig(t')) < (\sigma^-)^2(t') > - \\
&\int dt''(-ig(t''))^2 \left[< (\sigma^-(t'))^2(\sigma^-(t''))^2 > \phi(t'') - < (\sigma^-(t'))^2(\sigma^+(t''))^2 >\right]\phi(t'')\Big\} = 0\ .
\end{aligned}
$$

The expectation values are computed using Wick's theorem and using the free-field Green's functions of section II. The tadpole (time independent) is absorbed in a shift of the expectation value. The coefficient of $< \chi^+\chi^+ >, \quad < \chi^+\chi^- >$ must vanish independently because these Green's functions are independent and must vanish at all times. From the Green's functions (and more generally from the formal time contour integral) it is seen that the equations obtained are identical. If the expectation value is translational invariance, the spatial integrals set the momentum transfer to zero. For example using the zero temperature Green's functions of section I, one finds that the term that has the non-local (in time) correlation functions (last term) becomes

$$
\begin{aligned}
&\int dt''(ig(t''))^2 \left[< (\sigma^+(t'))^2(\sigma^+(t''))^2 > - < (\sigma^+(t'))^2(\sigma^-(t''))^2 >\right]\phi(t'') = \\
&\qquad 2\int dt''(ig(t''))^2\phi(t'')\Theta(t'-t'')\int \frac{d^3k}{(2\pi)^3}(-2i)\frac{\sin\left[2\omega_k(t'-t'')\right]}{4\omega_k^2}\ .
\end{aligned}
$$

For the resummed one-loop approximation the term $g\phi(t)(\sigma^\pm)^2$ is absorbed in the mass term for the σ field and only the $\chi\sigma^2$ terms are considered as perturbation. In this case the 1-loop contribution is obtained from the tadpole term $< (\sigma^\pm(t))^2 >$, which is now time dependent and obtained from the non-equilibrium Green's functions constructed with the mode functions for the time-dependent mass as in section IV. One now finds the evolution equations

$$
\begin{aligned}
&\ddot{\phi}(t) + m_\Phi^2\ \phi(t) + g(t) < \sigma^2(\vec{x},t) >= 0\ , \\
&< \sigma^2(\vec{x},t) >= -i\int \frac{d^3k}{(2\pi)^3}G^{++}_{k,\sigma}(t,t) = \int \frac{d^3k}{(2\pi)^3}\frac{|V_k(t)|^2}{2w_k}\coth\left[\frac{\beta w_k}{2}\right]\ , \\
&\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2(t)\right]V_k(t) = 0\ , \\
&V_k(0) = 1\ , \quad \dot{V}_k(0) = -iw_k = -i\sqrt{\vec{k}^2 + m^2(0)}\ , \\
&m^2(t) = m_\sigma + g(t)\phi(t)\ .
\end{aligned}
$$

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Figure Captions:

Fig.(1.a) Scalars: unbroken symmetry $\eta(\tau)$ vs τ for the values of the parameters $y = 0$; $\lambda/8\pi^2 = 0.2$; $g = \lambda = 1$; $m_\sigma = 0.2\, m_\phi$; $\eta(0) = 1.0$; $\dot{\eta}(0) = 0$.

Fig.(1.b): $\mathcal{N}_\sigma(\tau)$ vs. τ for the same value of the parameters as figure (1.a).

Fig.(1.c): $\mathcal{N}_{q,\sigma}(\tau = 120)$ vs. q for the same values as in figure (1.a).

Fig.(2.a) Scalars: broken symmetry $\eta(\tau)$ vs τ for the values of the parameters $y = 0$; $\lambda/8\pi^2 = 0.2$; $g = \lambda = 0.05$; $m_\sigma = 0.2\,|m_\phi|$; $\eta(0) = 0.6$; $\dot{\eta}(0) = 0$.

Fig.(2.b): $\mathcal{N}_\sigma(\tau)$ vs. τ for the same value of the parameters as figure (2.a).

Fig.(2.c): $\mathcal{N}_{q,\sigma}(\tau = 200)$ vs. q for the same values as in figure (2.a).

Fig.(3.a) Fermions: unbroken symmetry $\eta(\tau)$ vs τ for the values of the parameters $g = 0$; $y^2/\pi^2 = 0.5$; $\lambda/6y^2 = 1$; $m_\psi = 0$; $\eta(0) = 1.0$; $\dot{\eta}(0) = 0$.

Fig.(3.b): $\mathcal{N}^f(\tau)$ vs. τ for the same value of the parameters as figure (3.a).

Fig.(3.c): $\mathcal{N}^f_q(\tau = 200)$ vs. q for the same values as in figure (3.a).